MIDTERM SOLUTIONS - MATH 2121, FALL 2017.

Name:

Email:

Problem #	Max points possible	Actual score
1	15	
2	20	
3	15	
4	20	
5	20	
6	10	
Total	100	

You have **120 minutes** to complete this exam.

No books, notes, or electronic devices can be used on the test.

Clearly label your answers by putting them in a box.

Partial credit can be given on some problems if you show your work. Good luck!

Problem 1. (2 + 9 + 4 = 15 points)

(a) Write down the augmented matrix of the linear system

$$\begin{aligned} 2x_1 + x_3 + 21x_4 &= 5 \\ x_1 + 9x_2 + 8x_3 + 3x_4 &= 6 \\ 3x_1 - 12x_4 &= 0. \end{aligned}$$

Solution.

$$\boxed{\begin{bmatrix} 2 & 0 & 1 & 21 & 5 \\ 1 & 9 & 8 & 3 & 6 \\ 3 & 0 & 0 & -12 & 0 \end{bmatrix}}$$

(b) Compute the reduced echelon form of the matrix in (a).

Solution.

We will need the fact that

$$7-8\cdot 29 = 7-8(30-1) = 7-240+8 = 15-240 = -225.$$

Also observe that

$$-225 = 9(-25).$$

We now compute

$$\begin{bmatrix} 2 & 0 & 1 & 21 & 5 \\ 1 & 9 & 8 & 3 & 6 \\ 3 & 0 & 0 & -12 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 0 & -12 & 0 \\ 2 & 0 & 1 & 21 & 5 \\ 1 & 9 & 8 & 3 & 6 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & -4 & 0 \\ 2 & 0 & 1 & 21 & 5 \\ 1 & 9 & 8 & 3 & 6 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 29 & 5 \\ 0 & 9 & 8 & 7 & 6 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 29 & 5 \\ 0 & 9 & 0 & 7-8\cdot 29 & 6-8\cdot 5 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 29 & 5 \\ 0 & 9 & 0 & -225 & -34 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 29 & 5 \\ 0 & 1 & 0 & -25 & -34/9 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & -25 & -34/9 \\ 0 & 0 & 1 & 29 & 5 \end{bmatrix}.$$

The last matrix is in reduced echelon form so the answer is

$$\boxed{\begin{bmatrix} 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & -25 & -34/9 \\ 0 & 0 & 1 & 29 & 5 \end{bmatrix}}$$

(c) How many solutions does our linear system

$$\begin{aligned} 2x_1 + x_3 + 21x_4 &= 5 \\ x_1 + 9x_2 + 8x_3 + 3x_3 &= 6 \\ 3x_1 - 12x_4 &= 0 \end{aligned}$$

have? To receive full credit, explain how you derive your answer.

Solution.

The previous part shows that the pivot positions of the augmented matrix of this linear system are $(1, 1)$, $(2, 2)$, and $(3, 3)$.

No pivot positions occur in the last column, so the linear system is consistent.

Since the fourth column is not a pivot column, x_4 is a free variable, so the linear system has $\boxed{\text{infinitely many solutions}}$.

Problem 2. (2 + 2 + . . . + 2 = 20 points) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Let A be a matrix. Indicate which of the following is TRUE or FALSE.

(1) If T is linear then its range is equal to its codomain.
(2) If T is linear and one-to-one then $n \geq m$.
(3) If T is linear and onto then $n \geq m$.
(4) If T is linear and invertible then $n \geq m$.
(5) If T is linear and invertible, then its inverse is linear and invertible.
(6) If T is linear, and A is its standard matrix, then A has size $n \times m$.
(7) If T is linear, and A is its standard matrix, and $T(v) = 0$ for a nonzero vector $v \in \mathbb{R}^n$, then the columns of A are not linearly independent.
(8) If B is a matrix with the same size as A and $Av = Bv$ whenever v is a vector such that the products Av and Bv are both defined, then $A = B$.
(9) If A is not invertible, then AB is not the identity matrix for any matrix B.
(10) If T is linear, and A is its standard matrix, and the range of T is not all of $\mathbb{R}^m$, then not every column of A is a pivot column.

Each part will be graded as follows: 0 points for a wrong answer, 1 point for no answer, 2 points for the correct answer. Explanations are not required for answers.

Solution:

(1) FALSE

Suppose $T(v) = 0$ for all v.
Then T is linear but its range is $\{0\}$ and its codomain is $\mathbb{R}^m$.

(2) FALSE

If T is linear and one-to-one then $n \leq m$.

(3) TRUE

(4) TRUE

If T is linear and invertible then $n = m$ which also means that $n \geq m$.

(5) TRUE

(6) FALSE

If T is linear and invertible then its standard matrix has size $m \times n$.

(7) TRUE

(8) TRUE

We can conclude that $A = B$ if just $Ae_i = Be_i$ for $i = 1, 2, \ldots, n$ since this implies that the two matrices have the same columns.

(9) FALSE

Consider $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
Neither of the matrices on the left are invertible since they are not square.

(10) FALSE

The correct statement would be "If T is linear, and A is its standard matrix, and the range of T is not all of $\mathbb{R}^m$, then not every ROW of A CONTAINS A PIVOT POSITION."

If $n = 1$ and $m = 2$ and $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then the range of T would not be all of $\mathbb{R}^m$, but every column of A is a pivot column.

Problem 3. (5 + 10 = 15 points)

(a) For what values of x is the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & x & 0 \\ -1 & 5 & 0 & 8 \\ 1 & 2 & 2 & 4 \end{bmatrix}$$

invertible?

Solution.

One solution uses the determinant.

We have

$$\det A = \det \begin{bmatrix} 3 & x & 0 \\ 5 & 0 & 8 \\ 2 & 2 & 4 \end{bmatrix} + 2 \det \begin{bmatrix} 0 & 3 & 0 \\ -1 & 5 & 8 \\ 1 & 2 & 4 \end{bmatrix}.$$

The first determinant is

$$\det \begin{bmatrix} 3 & x & 0 \\ 5 & 0 & 8 \\ 2 & 2 & 4 \end{bmatrix} = 3(0-16) - x(20-16) + 0 = -48 - 4x.$$

The second determinant is

$$\det \begin{bmatrix} 0 & 3 & 0 \\ -1 & 5 & 8 \\ 1 & 2 & 4 \end{bmatrix} = 0 - 3(-4-8) + 0 = 36.$$

So $\det A = -48 - 4x + 2(36) = (72 - 48) - 4x = 24 - 4x$.

A is invertible if and only if $\det A \neq 0$

We have $24 - 4x \neq 0$ if and only if $x \neq 6$.

So the answer is: $\boxed{A \text{ is invertible if } x \neq 6.}$

(b) Assuming x is such that

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & x & 0 \\ -1 & 5 & 0 & 8 \\ 1 & 2 & 2 & 4 \end{bmatrix}$$

is invertible, derive a formula for A^{-1}.

Solution.

Let $y = \frac{1}{x-6}$.

To compute A^{-1} we row reduce

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & x & 0 & 0 & 1 & 0 & 0 \\ -1 & 5 & 0 & 8 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ -1 & 5 & 0 & 8 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 4 & 0 & 0 & 0 & 1 \\ 0 & 3 & x & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 5 & 2 & 8 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 4 & -1 & 0 & 0 & 1 \\ 0 & 3 & x & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 3 & 0 & 1 & -2 \\ 0 & 2 & 0 & 4 & -1 & 0 & 0 & 1 \\ 0 & 3 & x & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 3 & 0 & 1 & -2 \\ 0 & 0 & -4 & 4 & -7 & 0 & -2 & 5 \\ 0 & 0 & x-6 & 0 & -9 & 1 & -3 & 6 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1+18y & -2y & 6y & -12y \\ 0 & 1 & 0 & 0 & 3+18y & -2y & 1+6y & -2-12y \\ 0 & 0 & 0 & 4 & -7-36y & 4y & -2-12y & 5+24y \\ 0 & 0 & x-6 & 0 & -9 & 1 & -3 & 6 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1+18y & -2y & 6y & -12y \\ 0 & 1 & 0 & 0 & 3+18y & -2y & 1+6y & -2-12y \\ 0 & 0 & x-6 & 0 & -9 & 1 & -3 & 6 \\ 0 & 0 & 0 & 4 & -7-36y & 4y & -2-12y & 5+24y \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1+18y & -2y & 6y & -12y \\ 0 & 1 & 0 & 0 & 3+18y & -2y & 1+6y & -2-12y \\ 0 & 0 & 1 & 0 & -9y & y & -3y & 6y \\ 0 & 0 & 0 & 1 & -7/4-9y & y & -1/2-3y & 5/4+6y \end{array}\right].$$

Therefore

$$A^{-1} = \begin{bmatrix} 1+18y & -2y & 6y & -12y \\ 3+18y & -2y & 1+6y & -2-12y \\ -9y & y & -3y & 6y \\ -7/4-9y & y & -1/2-3y & 5/4+6y \end{bmatrix}$$

$$= \frac{1}{4y} \begin{bmatrix} 4/y+72 & -8 & 24 & -48 \\ 12/y+72 & -8 & 4/y+24 & -8/y-48 \\ -36 & 4 & -12 & 24 \\ -7/y-36 & 4 & -2/y-12 & 5/y+24 \end{bmatrix}$$

$$= \frac{1}{4(x-6)} \begin{bmatrix} 4x-24+72 & -8 & 24 & -48 \\ 12x-72+72 & -8 & 4x-24+24 & -8x+48-48 \\ -36 & 4 & -12 & 24 \\ -7x+42-36 & 4 & -2x+12-12 & 5x-30+24 \end{bmatrix}$$

$$= \frac{1}{4(x-6)} \begin{bmatrix} 4x+48 & -8 & 24 & -48 \\ 12x & -8 & 4x & -8x \\ -36 & 4 & -12 & 24 \\ -7x+6 & 4 & -2x & 5x-6 \end{bmatrix}$$

Our final answer is

$$\boxed{A^{-1} = \frac{1}{4(x-6)} \begin{bmatrix} 4x+48 & -8 & 24 & -48 \\ 12x & -8 & 4x & -8x \\ -36 & 4 & -12 & 24 \\ -7x+6 & 4 & -2x & 5x-6 \end{bmatrix}}.$$

Problem 4. (1 + 9 + 4 + 4 + 2 = 20 points) Consider the matrix

$$A = \begin{bmatrix} 5 & -1 & 6 & -4 \\ 2 & 3 & 16 & -5 \\ 0 & 2 & 8 & -2 \end{bmatrix}.$$

Remember that

- The *column space* of A is the span of its columns.
- The *null space* of A is the set of vectors v with $Av = 0$.

(a) What are the values of m and n such that $\mathrm{Col}A \subset \mathbb{R}^m$ and $\mathrm{Nul}A \subset \mathbb{R}^n$?

Solution.

$$\boxed{m = 3 \text{ and } n = 4}$$

(b) Compute the reduced echelon form of

$$A = \begin{bmatrix} 5 & -1 & 6 & -4 \\ 2 & 3 & 16 & -5 \\ 0 & 2 & 8 & -2 \end{bmatrix}.$$

Solution.

$$\begin{bmatrix} 5 & -1 & 6 & -4 \\ 2 & 3 & 16 & -5 \\ 0 & 2 & 8 & -2 \end{bmatrix} \to \begin{bmatrix} 5 & -1 & 6 & -4 \\ 2 & 3 & 16 & -5 \\ 0 & 1 & 4 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 5 & 0 & 10 & -5 \\ 2 & 3 & 16 & -5 \\ 0 & 1 & 4 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 3 & 16 & -5 \\ 0 & 1 & 4 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 3 & 12 & -3 \\ 0 & 1 & 4 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 1 & 4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathtt{RREF}(A)$$

(c) Find a basis for the column space of

$$A = \begin{bmatrix} 5 & -1 & 6 & -4 \\ 2 & 3 & 16 & -5 \\ 0 & 2 & 8 & -2 \end{bmatrix}.$$

Solution.

The previous part shows that the pivot columns of A are the first and second columns.

These columns are therefore a basis $\boxed{\left\{ \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \right\}}$.

(d) Find a basis for the null space of

$$A = \begin{bmatrix} 5 & -1 & 6 & -4 \\ 2 & 3 & 16 & -5 \\ 0 & 2 & 8 & -2 \end{bmatrix}.$$

Solution.

Since

$$\texttt{RREF}(A) = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

it follows that $Ax = 0$ if and only if

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

where

$$x_1 + 2x_3 - x_4 = x_2 + 4x_3 - x_4 = 0.$$

This means x belongs to the null space of A if and only if x has the form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 + x_4 \\ -4x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors $\boxed{\left\{ \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}}$ are therefore a basis for the null space.

(e) What are the dimensions of the column space and null space of A?

Solution.

Both subspaces have dimension 2.

Problem 5. (5 + 5 + 5 + 5 = 20 points)

(a) Does there exist a 3×3 matrix whose column space contains

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ but not } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}?$$

If there is, give an example. If there isn't, explain why not.

Solution.

The third vector is not a linear combination of the first two since the first two both belong to the null space of the 1-by-3 matrix

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

while the third does not. The matrix

$$\boxed{\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ -3 & -1 & 0 \end{bmatrix}}$$

is therefore a solution: its column space is exactly the span of the first two vectors, which does not contain the third.

(b) Does there exist a 3×3 matrix whose null space contains

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ but not } \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}?$$

If there is, give an example. If there isn't, explain why not.

Solution.

Label the vectors as u, v, w respectively. Then

$$u + w = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = 2v$$

so $w = -u + 2v$ is a linear combination of the first two vectors. Therefore any subspace containing the first two vectors must also contain the third. Since the null space of a matrix is a subspace, $\boxed{\text{no 3-by-3 matrix exists}}$ which contains the first two vectors but not the third.

(c) Does there exist a 3×3 matrix whose null space *and* column space contains

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}?$$

If there is, give an example. If there isn't, explain why not.

Solution.

The null space and column space of the matrix

$$\boxed{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -3 & -3 & -3 \end{bmatrix}}$$

contains the given vector.

(d) Does there exist a 3×3 matrix whose null space *and* column space contains

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}?$$

If there is, give an example. If there isn't, explain why not.

Solution.

These vectors are linearly independent since neither is a scalar multiple of the other. Therefore, if the null space and column space of a 3-by-3 matrix contained both vectors, then the dimensions of both subspaces would be at least 2. But this is impossible since the sum of these dimensions is 3 by the Rank Theorem. Therefore $\boxed{\text{no such matrix exists}}$.

Problem 6. (10 points) Find all 2×2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that $2a$ is a positive integer, b, c, d are real numbers, and $A^T = A^{-1}$.

Hint: for any square matrix B, recall that $\det(BB^T) = \det(B)\det(B^T) = \det(B)^2$.

Solution.

Suppose A is invertible and $A^{-1} = A^T$. Then, by the hint, we have

$$\det(A)^2 = \det(AA^T) = \det(AA^{-1}) = \det(I) = 1$$

so $\det(A) = ad - bc = \pm 1$. By assumption

(*) $$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}.$$

There are two cases, according to whether $\det A = 1$ or $\det A = -1$.

First suppose $\det A = 1$. Then (*) implies $a = d$ and $b = -c$, so

$$ad - bc = a^2 + b^2 = 1$$

and

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Thus (a, b) is a point of the unit circle, so $-1 \leq a \leq 1$ and if $2a$ is to be a positive integer then either $2a = 1$ or $2a = 2$. In the first case $a = 1/2$ and $b = \pm\sqrt{3}/2$, while in the second case $a = 1$ and $b = 0$. Thus, when $\det A = 1$, there are exactly three matrices with the given properties:

$$\boxed{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}}.$$

Next suppose $\det A = -1$. Then (*) implies that $a = -d$ and $b = c$, so

$$ad - bc = -a^2 - b^2 = -1$$

and

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}.$$

In this case (a, b) is again a point of the unit circle, so either $2a = 1$ or $2a = 2$. As before, if $2a = 1$ then $a = 1/2$ and $b = \pm\sqrt{3}/2$ while if $2a = 2$ then $a = 1$ and $b = 0$. Thus, when $\det A = -1$, there are three more matrices with the given properties:

$$\boxed{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}}.$$

These 6 matrices give all the ones with the given properties.