Note: As with the practice midterm, the following list of problems is longer than what will appear on the actual final. Some problems may also turn out to be more difficult than the problems you'll see on the exam. On average, however, these problems should be fairly similar in difficulty to the exam problems, and they cover most of the material that you should review.

These exercises focus more on the second half of the course.

For problems related to the first half, see the practice midterm.

1. Give the definitions of (a) *vector space*, (b) *subspace* of a vector space, and (c) *linear transformation* between vector spaces

2. Let V be the set of polynomials $f(x)$ in one variable of degree at most 3.

   This means that $x^3 + x \in V$ and $x^2 - 4 \in V$ but $x^4 \notin V$.

   Let D be the subset of polynomials $f(x) \in V$ with $f(0) = 0$.

   Let E be the subset of polynomials $f(x) \in V$ with $f(1) = 0$.

   (a) Explain when V is a vector space.

   (b) Give a basis for V. What is $\dim V$?

   (c) Explain why D and E are subspaces of V.

   (d) Give a basis for D. What is $\dim D$?

   (e) Find an invertible linear function $T : D \to E$.

   (f) Use the previous two parts to find a basis for E. What is $\dim E$?

3. Give the definitions of (a) *eigenvector*, (b) *eigenvalue*, and (c) *diagonalisable*.

4. Consider the matrix

$$A = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}.$$

   (a) Find the eigenvalues of A. Do this without using a calculator.

   (b) Find a basis for each eigenspace of A.

   (c) Is A diagonalisable? If it is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Then find an exact formula for A^n for any n.

5. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (a) Is A invertible? Explain why or why not.

   (b) Is A diagonalisable? Explain why or why not.

   (c) Find an exact formula for A^n for any positive integer n.

6. Find examples of the following:
   (a) A matrix which is not invertible or diagonalisable.
   (b) A matrix which is symmetric but not invertible.
   (c) A matrix which is not diagonal or invertible, but is diagonalisable.
   (d) A 3×3 matrix which is diagonalisable but not diagonal, with only two eigenvalues.
   (e) A 3×3 matrix with all real entries and two complex eigenvalues which are not in $\mathbb{R}$.

7. Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that
$$\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix} = PCP^{-1}.$$

8. Find an exact formula for the nth term for the sequence a_n which begins as $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, and satisfies $a_{n+3} = a_n + a_{n+1} + a_{n+2}$ for $n \geq 0$.

9. Give definitions of $u \bullet v$ and $\|v\|$ for vectors $u, v \in \mathbb{R}^n$. What is a *unit vector*? Define what it means for a set of vectors to be *orthogonal* and *orthonormal*.

10. Find an orthonormal basis for the column space of the matrix
$$A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 3 & 5 & 8 \\ 1 & -1 & -3 & -2 \end{bmatrix}.$$
Then find an orthonormal basis for $(\mathrm{Col}\, A)^{\perp}$.

11. Give a formula for the orthogonal projection of a vector $y \in \mathbb{R}^3$ onto the plane
$$H = \{v \in \mathbb{R}^3 : v_1 + 2v_2 + 3v_3 = 0\}.$$

12. Find the best approximation to z by vectors of the form $c_1v_1 + c_2v_2$ when
$$z = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}, \quad \text{and} \quad v_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}.$$

13. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ has the following values:

x	$f(x)$
0	0
1	6
2	5
3	10
4	7

(a) Find the equation of the line $y = ax + b$ that best approximates $f(x)$ is the sense of least-squares.

(b) Find the equation of the parabola $y = ax^2 + bx + c$ that best approximates $f(x)$ is the sense of least-squares.

(c) How would you find a function of the form $g(x) = 2^{ax+b}$ that is a good approximation for $f(x)$?

14. Consider the symmetric matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}.$$

Find an orthogonal matrix U (that is, an invertible matrix with $U^T = U^{-1}$) and a diagonal matrix D such that

$$A = UDU^T.$$

Repeat this exercise with 1, 2, 3, 4, 5, 6 replaced by a random list of six numbers.

15. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

16. Do the first problem in each section of supplementary exercises for Chapters 1-7.