# 1 Last time: linear systems and row operations

Here's what we did last time: a *system of linear equations* or *linear system* is a list of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where each $x_i$ is a variable and each $a_{ij}, b_j$ for $1 \leq i, j \leq m$ is a number.

The *coefficient matrix* and *augmented matrix* of such a system are respectively

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \\ a_{m1} & \dots & a_{mn} & b_m \end{bmatrix}.$$

The coefficient matrix is $m \times n$: it has $m$ rows and $n$ columns.

The augmented matrix has one extra column, so has size $m \times (n+1)$.

A *solution* to a linear system is a list of numbers $(s_1, s_2, \dots, s_n)$ such that setting $x_1 = s_1$, $x_2 = s_2$, $\dots$, $x_n = s_n$, all at the same time, makes each equation in the system a true statement. Two linear systems are *equivalent* if they have the same solutions.

Important fact: Any linear system has either 0, 1, or infinitely many solutions.

We solve a linear system by performing *row operations* on its augmented matrix.

The following are row operations:

(1) Replace one row by the sum of itself and a multiple of another row.

(2) Multiply all entries in one row by a fixed nonzero number.

(3) Interchange two rows.

Let's do an example to see these rules in action.

**Example.** Consider the linear system

$$\begin{aligned} x_1 + 2x_2 + 5x_3 &= 1 \\ x_1 + x_3 &= 0 \\ x_2 + x_3 &= 7 \end{aligned} \quad \text{which has augmented matrix} \quad \begin{bmatrix} 1 & 2 & 5 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix}$$

Adding $-1$ times the second row to the first is an example of row operation (1):

$$\begin{bmatrix} 1 & 2 & 5 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{4} & \mathbf{1} \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix}.$$

Next let's add $-2$ times the last row to the first row:

$$\begin{bmatrix} 0 & 2 & 4 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{-13} \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix}.$$

Now lets use rule (3) to swap some rows:

$$\begin{bmatrix} 0 & 0 & 2 & -13 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} 0 & 1 & 1 & 7 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -13 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 2 & -13 \end{bmatrix}.$$

Now lets scale the third row by 1/2:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 2 & -13 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-13/2} \end{bmatrix}.$$

Finally, lets use (1) twice to cancel the entries in rows 1 and 2 in column 3:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & -13/2 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{13/2} \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & -13/2 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 & 0 & 13/2 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{27/2} \\ 0 & 0 & 1 & -13/2 \end{bmatrix}.$$

Two linear systems are *row equivalent* if their augmented matrices can be transformed to each other by a sequence of row operations.

**Theorem.** Row equivalent linear systems are equivalent, i.e., have the same solutions.

Therefore the original system in our example has the same solutions as the system corresponding to the last matrix, which is just

$$\begin{aligned} x_1 &= 13/2 \\ x_2 &= 27/2 \\ x_3 &= -13/2. \end{aligned}$$

This system has unique solution $(13/2, 27/2, -13/2)$, so the original system also has one solution.

## 2 Row reduction to echelon form

The goal today is to give an algorithm to determine whether a linear system has 0, 1, or $\infty$ solutions, and to determine what these solutions are. The algorithm will be called "row reduction to echelon form" and will formalize the way we solved the linear system in the last example. Sometimes, this algorithm is also called "Gaussian elimination."

A row in a matrix

$$\begin{bmatrix} a & b & \dots & z \end{bmatrix}$$

is *nonzero* if not every entry in the row is zero.

A column in a matrix

$$\begin{bmatrix} a \\ b \\ \vdots \\ z \end{bmatrix}$$

is *nonzero* if not every entry in the column is zero.

The *leading entry* in a row of a matrix is the first nonzero entry from left going right. For example,

$$\begin{bmatrix} 0 & 0 & 7 & 0 & 5 \end{bmatrix}$$

has leading entry 7. The leading entry occurs in the 3rd column.

**Definition.** A matrix with $m$ rows and $n$ columns is in *echelon form* if it has the following properties:

1. If a row is nonzero, then every row above it is also nonzero.
2. The leading entry in a nonzero row is in a column to the right of the leading entry in the row above.
3. If a row is nonzero, then every entry below its leading entry in the same column is zero.

It is easier to understand this definition through an example than the list of conditions:

$$\begin{bmatrix} 0 & 0 & 5 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 6 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

is in echelon form. Here each * can be replaced by an arbitrary number.

The matrix

$$\begin{bmatrix} 0 & 0 & 5 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 6 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is also in echelon form.

The matrix

$$\begin{bmatrix} 0 & 0 & 0 & 6 & * & * & * & * & * & * \\ 0 & 0 & 5 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

is **not** in echelon form: the leading entry in the first row is left of the leading entry is the second row.

The matrix

$$\begin{bmatrix} 0 & 0 & 6 & * & * & * & * & * & * & * \\ 0 & 0 & 5 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

is **not** in echelon form: nonzero entries in same column below the leading entry 6 in the first row.

There is a slightly more restrictive form of echelon form which will be useful.

**Definition.** A matrix is in *reduced echelon form* if

1. The matrix is in echelon form.
2. Each nonzero row has leading entry 1.
3. The leading 1 in each nonzero row is the only nonzero number in its column.

The matrix in echelon form

$$\begin{bmatrix} 0 & 0 & 5 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 6 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

is not reduced. But we can apply row operations to turn it into reduced echelon form:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The fundamental theorem of today is the following:

**Theorem.** Each matrix $A$ is row equivalent to exactly one matrix $U$ in reduced echelon form.

The proof of this result is not super hard, but is easier to describe using terminology we will introduce in later lectures. The textbook has a proof in its appendix. By the end of today, the result should at least seem plausible, once we understand how to construct the matrix $U$ from $A$.

We call $U$ the *(row) reduced echelon form* of $A$. We often denote this $\texttt{RREF(A)}$.

A matrix $E$ in echelon which is row equivalent to $A$ is called an *echelon form* of $A$.

**Proposition.** In any echelon form $E$ of a matrix $A$, the locations of the leading entries are the same.

*Proof.* Row operations which transform $E$ to the unique reduced echelon form of $A$ do not change these locations: these row operations always just rescale a row, or add a nonzero row to a row above it. □

A *pivot position* in a matrix $A$ is the location containing a leading 1 in the reduced echelon form for $A$.

A *pivot column* in a matrix$A$ is a column contining a pivot position.

**Example.** Suppose

$$A = \begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & 1 & 1 \end{bmatrix}.$$

Let's find $\texttt{RREF(A)}$. Add $-1$ times first row to third row, then $-2/3$ times first row to second row:

$$A = \begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & \boxed{3} & 1 & 0 \\ 0 & 0 & \boxed{-2/3} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}.$$

The last matrix is in echelon form, but is not reduced. Pivot positions are boxed. To get to reduced echelon form, rescale rows 1 and 2 by $1/3$ and $-3/2$, then add multiple of second row to first:

$$\begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 0 & -2/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \texttt{RREF(A)}.$$

Columns 2, 3, and 4 are the pivot columns.

We sometimes refer to an entry in a pivot position of a matrix as a *pivot*.

**Algorithm** (Row reduced to echelon form)**.**

Input: an $m \times n$ matrix $A$.

Procedure:

1. Begin with the leftmost nonzero column.

   This is a pivot column. The pivot position is the top position of the column.

2. Select a nonzero entry in the current pivot column.

   If needed, perform a row operation to swap the row with this entry and the top row.

3. Use row operations to create zeros in the entries below the pivot position.

4. Cover the row containing the current pivot position, and then apply the previous steps to the $(m-1) \times n$ submatrix that remains. Repeat until the entire matrix is in echelon form.

5. Start with the row containing the rightmost pivot position in our matrix, now in echelon form.

   Use row operations to rescale this row to have leading entry 1.

   Then use row operations to create zeros in the entries in the same column above each leading entry.

Repeat this for each successive pivot position going left, until the matrix is in reduced echelon form.

Output: `RREF(A)`.

Here is a detailed example showing each step.

**Example.** Consider the input

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 \\ 3 & -7 & 8 & -5 \\ 3 & -9 & 12 & -9 \end{bmatrix}.$$

1. Begin with the first column; the pivot position is boxed:

$$\begin{bmatrix} \boxed{\mathbf{0}} & 3 & -6 & 6 \\ \mathbf{3} & -7 & 8 & -5 \\ \mathbf{3} & -9 & 12 & -9 \end{bmatrix}.$$

2. Select the 3 in the second row of the first column. Swap rows 1 and 2:

$$\begin{bmatrix} \boxed{\mathbf{0}} & 3 & -6 & 6 \\ \mathbf{3} & -7 & 8 & -5 \\ \mathbf{3} & -9 & 12 & -9 \end{bmatrix} \to \begin{bmatrix} \boxed{3} & -7 & 8 & -5 \\ 0 & 3 & -6 & 6 \\ 3 & -9 & 12 & -9 \end{bmatrix}.$$

3. Use row operations to create zeros below the boxed pivot position:

$$\begin{bmatrix} \boxed{3} & -7 & 8 & -5 \\ 0 & 3 & -6 & 6 \\ 3 & -9 & 12 & -9 \end{bmatrix} \to \begin{bmatrix} \boxed{3} & -7 & 8 & -5 \\ 0 & 3 & -6 & 6 \\ 0 & -2 & 4 & -4 \end{bmatrix}$$

4. Repeat 1-3 on bottom right submatrix:

$$\left[\begin{array}{c|ccc} 3 & -7 & 8 & -5 \\ \hline 0 & \boxed{3} & -6 & 6 \\ 0 & -2 & 4 & -4 \end{array}\right] \to \left[\begin{array}{c|ccc} 3 & -7 & 8 & -5 \\ \hline 0 & \boxed{3} & -6 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{c|c|cc} 3 & -7 & 8 & -5 \\ \hline 0 & 3 & -6 & 6 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

5. We now have a matrix in echelon form, pivot positions in boxes:

$$\begin{bmatrix} \boxed{3} & -7 & 8 & -5 \\ 0 & \boxed{3} & -6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Recale rightmost pivot, then cancel entries above rightmost pivot position in same column:

$$\begin{bmatrix} 3 & -7 & 8 & -5 \\ 0 & \boxed{3} & -6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -7 & 8 & -5 \\ 0 & \boxed{1} & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & -6 & 9 \\ 0 & \boxed{1} & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Repeat with the next pivot position, going right to left:

$$\begin{bmatrix} \boxed{3} & 0 & -6 & 9 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 0 & -2 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The result is in reduced echelon form:

$$\texttt{RREF(A)} = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

# 3 Solutions of linear systems

**Example.** If a linear system has augmented matrix $A$ with

$$\texttt{RREF(A)} = \begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

then the system is equivalent to

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0. \end{aligned}$$

The pivot columns of $A$ are 1 and 2. Call $x_1$ and $x_2$ the *basic variables*.

The non-pivot column is 3. Call $x_3$ a *free variable*.

To find all solutions to the system, choose any values for the free variables and then solve for the basic variables. In the above system, we have $x_1 = 5x_3 + 1$ and $x_2 = 4 - x_3$.

Hence all solutions to this system have the form $(s_1, s_2, s_3) = (5a + 1, 4 - a, a)$ for $a \in \mathbb{R}$.

In general, consider a linear system whose augmented matrix $A$.

A variable $x_i$ is *basic* is $i$ is a pivot column, and *free* otherwise.

- The system has 0 solutions if the last column is a pivot column of $A$.

  In this case `RREF(A)` has a row of the form

  $$\begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

  so our system is equivalent to a linear system containing the false equation $0 = 1$.

- The system has $\infty$ solutions if the last column is not a pivot column but there is $\geq 1$ free variable.
- The system has 1 solution if there are no free variables, and the last column is not a pivot column.

Once we have computed `RREF(A)` and identified the free and basic variables, we can write down all solutions to the system (if there are solutions) exactly as in the example: by letting each free variable be arbitrary, and then solving for the basic variables in terms of the free variables.

Next time: vectors in $\mathbb{R}^n$.