1 Last time: linear transformations

<u>Notation</u>. Writing

$$f : X \to Y$$

means that f is a function which takes inputs from the set X and produces outputs in the set Y.

X is the *domain* of the function f.

Y is the *codomain* of f.

The *image* of an input x in X under f is the ouput $f(x)$.

The *image* or *range* of the function f is the set of possible outputs:

$$\text{range}(f) = \{f(x) : x \in X\}.$$

The range is contained in the codomain, but might be smaller than it.

Important definition. A function $f : \mathbb{R}^n \to \mathbb{R}^m$, with domain and codomain given by sets of vectors, is a *linear transformation* if

(i) $f(u + v) = f(u) + f(v)$ for all vectors $u, v \in \mathbb{R}^n$.

(ii) $f(cv) = cf(v)$ for all vectors $v \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$.

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then some other properties also hold:

$f(0) = 0$.

$f(u - v) = f(u) - f(v)$ for $u, v \in \mathbb{R}^n$.

$f(au + bv) = af(u) + bf(v)$ for all $a, b \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$.

$f(a_1v_1 + a_2v_2 + \cdots + a_mv_m) = a_1f(v_1) + a_2f(v_2) + \cdots + a_mf(v_m)$ for all $a_i \in \mathbb{R}$ and $v_i \in \mathbb{R}^n$.

Linear transformations are closely related to matrices, by the following statement:

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then there is a unique $m \times n$ matrix A such that $T(v) = Av$ for all $v \in \mathbb{R}^n$.

The matrix A is called the *standard matrix* of T, and is computed as

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}$$

where $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ are the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Example. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the function $T(v) = 4v$.

This is a linear transformation because

$$T(u + v) = 4(u + v) = 4u + 4v = T(u) + T(v) \qquad \text{and} \qquad T(cv) = 4(cv) = c(4v) = cT(v).$$

What is the standard matrix of T?

We have $T(e_1) = 4e_1$, $T(e_2) = 4e_2$, and $T(e_3) = 4e_3$. Therefore $T(v) = Av$ where

$$A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Write A_{ij} for the entry in the ith row and jth column of A. Then $A_{ij} = 0$ whenever $i \neq j$. We call a matrix with this property *diagonal*.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1^2 + v_2^2 + \dots + v_n^2.$$

This function is *not* linear: we have $T(2v) = 4T(v) \neq 2T(v)$ for any nonzero vector $v \in \mathbb{R}^n$.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix}.$$

This function is a linear transformation. (Why?) Its standard matrix is

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_{n-1}) & T(e_n) \end{bmatrix} = \begin{bmatrix} e_n & e_{n-1} & \dots & e_2 & e_1 \end{bmatrix} = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}.$$

In the matrix on the right, we adopt the convention of only writing the nonzero entries: all positions in the matrix which are blank contain zeros.

2 One-to-one and onto functions

This section talks about two important classes of linear transformations, which can be characterized in terms of whether the columns of the standard matrix are linearly independent or span the codomain.

Definition. A function $f : X \to Y$ is *one-to-one* or *injective* if $f(a) = f(b)$ implies $a = b$. In words: f does not send two different inputs to the same output. If $a \neq b$ and $f(a) = f(b)$ then f is *not one-to-one*.

Example. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear transformation $T(v) = Av$ where

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 5 & 3 \end{bmatrix}.$$

Is T one-to-one? No: since A has more columns than rows, its columns are linearly dependent. Therefore there is a vector $0 \neq v \in \mathbb{R}^3$ such that $T(v) = Av = 0$. But we also have $T(0) = 0$

Theorem. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then T is one-to-one if and only if the only solution to $T(x) = 0$ is $x = 0 \in \mathbb{R}^n$, i.e., the columns of the standard matrix A of T are linearly independent.

Proof. Suppose the only solution to $T(x) = 0$ is $x = 0 \in \mathbb{R}^n$. Then whenever $u, v \in \mathbb{R}^n$ are vectors with $u \neq v$, we have $T(u) - T(v) = T(u - v) \neq 0$ since $u - v \neq 0$, so $T(u) \neq T(v)$. Therefore T is one-to-one.

If T is one-to-one, then $T(x) = T(0) = 0$ implies $x = 0$, so $T(x) = 0$ has only trivial solutions. □

Definition. A function $f : X \to Y$ is *onto* or *surjective* if $\text{range}(f) = \{f(x) : x \in X\} = Y$. In words: the range of f is equal to its codomain. If there is a value $y \in Y$ such that $f(x) \neq y$ for all $x \in X$, then f is *not onto*.

Example. Suppose again that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear transformation $T(v) = Av$ where

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 5 & 3 \end{bmatrix}.$$

Is T onto? Yes: the columns of A span $\mathbb{R}^2$ if and only if A has a pivot position in every row, and we have

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3/5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 19/5 \\ 0 & 1 & 3/5 \end{bmatrix} = \texttt{RREF(A)}.$$

Theorem. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then T is onto if and only if the columns of the standard matrix A of T span $\mathbb{R}^m$.

Proof. The vectors in the range of T are precisely the linear combinations of the columns of A.

The range is $\mathbb{R}^m$ precisely when the span of the columns of A is $\mathbb{R}^m$. □

Example. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 3v_1 + v_2 \\ 5v_1 + 7v_2 \\ v_1 + 3v_2 \end{bmatrix}.$$

This function is a linear transformation. Its standard matrix is

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}.$$

To determine if T is one-to-one, we check if the columns of A linearly independent. To do this, we row reduce to echelon form:

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & -8 \\ 0 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \texttt{RREF(A)}.$$

This shows that A has a pivot position in every column, which means $Ax = 0$ has only trivial solutions, which means the columns of A are linearly independent, which means T is one-to-one.

To determine if T is onto, we want to find out if the columns of A span $\mathbb{R}^3$. From last time, we know that this happens if and only if A has a pivot position in every row. Since the third row of A has no pivot position, T is not onto.

Corollary. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one only if $n \leq m$, and onto only if $n \geq m$.

Proof. Results last time show that T is one-to-one iff its standard matrix has a pivot position in every column, and onto iff its standard matrix has a pivot position in every row. The first case requires there to be more columns n than rows m, and the second case requires there to be more rows m than columns n (since each row and each column contains at most one pivot position). □

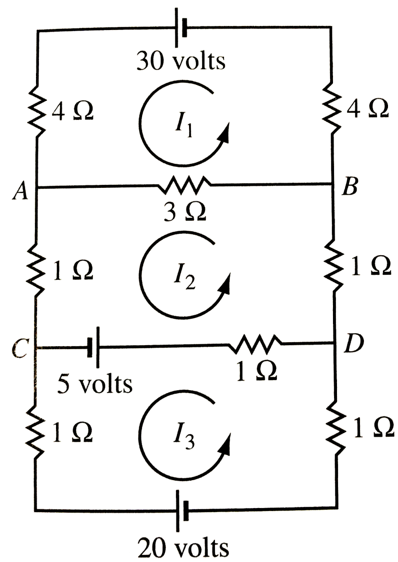
3 Geometric interpretations of linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with standard matrix A. We can illustrate T by drawing the parallelogram with sides $T(e_1)$ and $T(e_2)$. (Fill in these pictures yourself.)

Standard matrix of T	Picture	Description of T
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$		Reflect across the x-axis
$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$		Reflect across y-axis
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$		Reflect across $y = x$
$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ $(0 < k < 1)$		Horizontal contraction
$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ $(0 < k < 1)$		Vertical contraction
$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ $(k > 0)$		Horizontal sheering

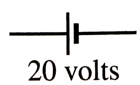
4 Application to electrical networks

An electrical circuit is represented by a diagram like this:



In this sort of picture, a symbol of the form

means a resistor with *resistance* 8 ohms, while a symbol of the form



means a *voltage* drop (in the direction going from the long edge to the short edge, in this case left to right) of 20 volts. Going from right to left across this component, the voltage rises by 20 volts.

Electrical current is measured in *amps* and flows through the circuit.

The units of voltage, resistance, and current and defined so that 1 volt = 1 ohm × 1 amp.

Analogy. To understand what voltage, resistance, and current actually mean, it's useful to have this informal analogy in mind. Think of the circuit as a system of rivers and electricity as water flowing through this system. Voltage drops correspond to waterfalls, places where gravity causes the water to flow faster. A resistor corresponds to a patch of boulders or tree branches stuck in the river, which cause the water to flow more slowly. Current corresponds to the volume of water that is flowing.

The amount of current flowing through any point of the circuit is determined by this law:

Ohm's law. The voltage drop between two points in the circuit is equal to the current through those two points times the effective resistance between the two points. If V is the voltage drop, I is the current, and R is the resistance, then we have $V = IR$.

Our picture indicates 3 loops in the circuit.

The current through each loop obeys this law:

Kirchhoff's voltage law. The sum of the voltage drops IR in one direction around a loop in the circuit is equal to the sum of the voltage sources in the same direction around the loop.

Let I_1, I_2, and I_3 be the amount of current flowing in each loop the circuit. We can use Ohm's and Kirchhoff's laws to derive a system of linear equations which these values must satisfy.

- Loop 1: By Ohm's law, the voltage drops across the 3 resistors are

  $$4I_1, \qquad 3(I_1 - I_2), \qquad \text{and} \qquad 4I_1.$$

  To explain this, note that the the currents flowing in loops 1 and 2 between points A and B are in opposite directions, so the net current (in the direction of loop 1) is $I_1 - I_2$, and the voltage drop between A and B is $3(I_1 - I_2)$.

  The sum of the voltage sources in loop 1 is +30 volts, so by Kirchhoff's law

  $$30 = 11I_1 - 3I_2.$$

- Loop 2: The voltage drops across the 4 resistors are

  $$1I_2, \qquad 1(I_2 - I_3), \qquad 1I_2, \qquad \text{and} \qquad 3(I_2 - I_1).$$

  The sum of the voltage sources around loop 2 is +5 volts, so by Kirchhoff's law

  $$5 = -3I_1 + 6I_2 - I_3.$$

- Loop 3: The voltage drops across the 3 resistors are

  $$1I_3, \qquad 1I_3, \qquad \text{and} \qquad 1(I_3 - I_2)$$

  while the sum of the voltage sources around loop 3 is $-20 - 5 = -25$ volts. Therefore

  $$-25 = -I_2 + 3I_3.$$

This shows that the loop currents satisfy the matrix equation

$$\begin{bmatrix} 11 & -3 & 0 \\ -3 & 6 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 5 \\ -25 \end{bmatrix}. \qquad (*)$$

To solve for the values of the current, row reduce the augmented matrix corresponding to this system:

$$\begin{aligned} A &= \begin{bmatrix} 11 & -3 & 0 & 30 \\ -3 & 6 & -1 & 5 \\ 0 & -1 & 3 & -25 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -2 & 1/3 & -5/3 \\ 0 & 1 & -3 & 25 \\ 11 & -3 & 0 & 30 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & -17/3 & 145/3 \\ 0 & 1 & -3 & 25 \\ 11 & 0 & -9 & 105 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & -17/3 & 145/3 \\ 0 & 1 & -3 & 25 \\ 0 & 0 & 160/3 & -1280/3 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & -17/3 & 145/3 \\ 0 & 1 & -3 & 25 \\ 0 & 0 & 1 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -8 \end{bmatrix} = \texttt{RREF(A)}. \end{aligned}$$

Thus our system (*) has a unique solution, given by $\boxed{I_1 = 3 \text{ amps}, I_2 = 1 \text{ amp, and } I_3 = -8 \text{ amps.}}$