1 Vector spaces

This course focuses on $\mathbb{R}^n$ and its subspaces.

These objects are examples of *(real) vector spaces*.

(There is also a notion of a *complex vector space* where our scalars can be complex numbers from $\mathbb{C}$ rather than just $\mathbb{R}$. Essentially all of the theory is the same, so for now we stick to real vector spaces which are more closely aligned with applications.)

The definition of a general vector space is given as follows:

Definition. A *vector space* is a nonempty set V with two operations called *vector addition* and *scalar multiplication* satisfying several conditions. We refer to the elements of V as *vectors*.

The vector addition operation for V should be a rule that takes two input vectors $u, v \in V$ and produces an output vector $u + v \in V$ such that

(a) $u + v = v + u$.

(b) $(u + v) + w = u + (v + w)$.

(c) There exists a unique *zero vector* $0 \in V$ with the property that $0 + v = v$ for all $v \in V$.

The scalar multiplication operation for V should be a rule that takes a scalar input $c \in \mathbb{R}$ and an input vector $v \in V$ and produces an output vector $cv \in V$ such that

(a) If $c = -1$ then $v + (-1)v = 0$.

(b) $c(u + v) = cu + cv$.

(c) $(c + d)v = cv + dv$ for $c, d \in \mathbb{R}$.

(d) $c(dv) = (cd)v$ for $c, d \in \mathbb{R}$.

(e) If $c = 1$ then $1v = v$.

Notation: If V is a vector space and $v \in V$ then we define $-v = (-1)v$ and $u - v = u + (-v)$.

Example. $\mathbb{R}^n$ and any subspace of $\mathbb{R}^n$ is a vector space, with the usual operations of vector addition and scalar multiplication.

Example. Let $\mathbb{R}^\infty$ be the set of infinite sequences $a = (a_1, a_2, a_3, \dots)$ of real numbers $a_i \in \mathbb{R}$. Define

$$a + b = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots) \qquad \text{and} \qquad ca = (ca_1, ca_2, ca_3, \dots)$$

for $a, b \in \mathbb{R}^\infty$ and $c \in \mathbb{R}$.

These operations make $\mathbb{R}^\infty$ into a vector space.

The zero vector in this space is the sequence $0 = (0, 0, 0, \dots) \in \mathbb{R}^\infty$.

It is rarely necessary to check the axioms of a vector space in detail, and not too useful to memorise the abstract definition. If we have a set with operations that look like vector addition and scalar multiplication for $\mathbb{R}^n$, then we usually have a vector space. Moreover, it's usually easy to identify any vector space we encounter as a special case of a few general constructions like the following:

Example. Let X be any set. Define $\mathrm{Map}(X, \mathbb{R})$ as the set of functions $f : X \to \mathbb{R}$.

Given $f, g \in \mathrm{Map}(X, \mathbb{R})$ define $f + g$ as the function with the formula

$$(f + g)(x) = f(x) + g(x) \qquad \text{for } x \in X.$$

Given $c \in \mathbb{R}$ and $f \in \mathrm{Map}(X, \mathbb{R})$, define cf as the function with the formula

$$(cf)(x) = cf(x) \qquad \text{for } x \in X.$$

The set $\mathrm{Map}(X, \mathbb{R})$ is a vector space relative to these operations. The corresponding zero vector in $\mathrm{Map}(X, \mathbb{R})$ is the function $f(x) = 0$.

In a sense which can be made precise, we have

$\mathbb{R}^n = \mathrm{Map}(\{1, 2, 3, \dots, n\}, \mathbb{R})$.

$\mathbb{R}^\infty = \mathrm{Map}(\{1, 2, 3, \dots\}, \mathbb{R})$.

More generally, if V is any vector space then the set of functions $\mathrm{Map}(X, V) = \{f : X \to V\}$ is a vector space for similar definitions of vector addition and scalar multiplication.

As an example of how one can use the axioms to prove properties of a general vector space, consider the following identities which are obvious for subspaces of $\mathbb{R}^n$.

Proposition. If V is a vector space then $0v = 0$ and $c0 = 0$ for all $c \in \mathbb{R}$ and $v \in V$.

Proof. We have $0v = (0 + 0)v = 0v + 0v$ so $0 = 0v - 0v = (0v + 0v) - 0v = 0v + (0v - 0v) = 0v + 0 = 0v$.

Similarly, $c0 = c(0 + 0) = c0 + c0$ so $0 = c0 - c0 = (c0 + c0) - c0 = c0 + (c0 - c0) = c0 + 0 = c0$. □

2 Subspaces, bases, and dimension

Definition. A *subspace* of a vector space V is a subset H containing the zero vector of V, such that if $u, v \in H$ then $u + v \in H$ and if $c \in \mathbb{R}$ and $v \in H$ then $cv \in H$.

If $H \subset V$ is a subspace then H is itself a vector space with the same operations of scalar multiplication and vector addition.

Example. V is a subspace of itself and $\{0\} \subset V$ is a subspace.

Example. $\mathbb{R}^2$ is technically not a subspace of $\mathbb{R}^3$ since $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$.

Example. Let X be any set. Let $Y \subset X$ be a subset. Define H as the subset of $\mathrm{Map}(X, \mathbb{R})$ consists of the functions $f : X \to \mathbb{R}$ with $f(y) = 0$ for all $y \in Y$. Then H is a subspace.

Example. The set of all functions $\mathrm{Map}(\mathbb{R}^n, \mathbb{R}^m)$ is a vector space since $\mathbb{R}^m$ is a vector space. The subset of linear functions $f : \mathbb{R}^n \to \mathbb{R}^m$ is a subspace of this vector space.

Let V be a vector space.

A *linear combination* of a finite list of vectors $v_1, v_2, \dots, v_k \in V$ is a vector of the form $c_1v_1 + c_2v_2 + \dots + c_kv_k$ for some scalars $c_1, c_2, \dots, c_k \in \mathbb{R}$. A linear combination by definition only involves finitely many vectors.

The *span* of a set of vectors is the set of all linear combinations that can be formed from the vectors. It is important to note that each such linear combination can only involve finitely many vectors at a time. The span of a set of vectors in V is a subspace of V.

Example. Let $V = \mathrm{Map}(\mathbb{R}, \mathbb{R})$. The span of the infinite set of functions $1, x, x^2, x^3, \dots \in V$ is the subspace $P \subset H$ of *polynomial functions*. Note that each polynomial function is a linear combination of a finite number of monomials $c_nx^n + c_{n-1}x^{n-1} + \dots + c_1x + c_0$.

A finite list of vectors $v_1, v_2, \ldots, v_k \in V$ is *linearly independent* if it is impossible to express $0 = c_1v_1 + c_2v_2 + \cdots + c_kv_k$ except when $c_1 = c_2 = \cdots = c_k = 0$. An infinite list of vectors is linearly independent if every finite subset is linearly independent.

Finally, a *basis* of a vector space V is a subset of linearly independent vectors whose span is V. Saying $b_1, b_2, b_3, \ldots$ is a basis for V is the same as saying that each $v \in V$ can be expressed as a uniquely linear combination of basis elements.

When V has a basis which is finite in size, then the notions of linear combinations, span, and linear independence work out exactly the same as our earlier definitions for vectors in $\mathbb{R}^n$. When V has no finite basis, for example in the case when $V = \mathbb{R}^\infty$, things are more complicated. The following properties still hold, but their proofs in general are a bit beyond the scope of this course:

Theorem. Let V be a vector space.

1. V has at least one basis.
2. Every basis of V has the same size.
3. If A is a subset of linearly independent vectors in V then V has a basis B with $A \subset B$.
4. If C is a subset of vectors in V whose span is V then V has a basis B with $B \subset C$.

As for subspaces of $\mathbb{R}^n$, we define the *dimension* of a vector space V to be the common size of any of its bases. Denote the dimension of V by $\dim V$.

Corollary. If $H \subset V$ is a subspace then $\dim H \leq \dim V$, and if $\dim H = \dim V$ then $H = V$.

Proof. This follows from the last two parts of the previous theorem. □

Example. If X is a finite set then $\dim \mathrm{Map}(X, \mathbb{R}) = |X|$ where $|X|$ is the size of X. A basis is given by the functions $\delta_y : X \to \mathbb{R}$ for $y \in X$, defined by the formulas

$$\delta_y(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

Example. Let $e_i \in \mathbb{R}^\infty$ be the infinite sequence with 1 in entry i and 0 is all other entries. The sequences $e_1, e_2, e_3, \ldots$ are linearly independent in $\mathbb{R}^\infty$ by essentially the same argument as used to show that the standard basis elements of $\mathbb{R}^n$ are linearly independent. This set of vectors is not a basis, since its span does not contain sequences like the constant sequence $(1, 1, 1, 1, \ldots)$, for example. (Instead, these sequences are a basis for the subspace of $\mathbb{R}^\infty$ of infinite sequences whose terms are eventually zero.)

Hard problem: can you describe a basis of $\mathbb{R}^\infty$?

The following is a more interesting example involving the space of solutions of a differential equation. The problem of of describing all solutions to a differential equation is an important motivation for the consideration of abstract vectors spaces (rather than just subspaces of $\mathbb{R}^n$) in the first place.

Example. Let V be the subset of $\mathrm{Map}(\mathbb{R}, \mathbb{R})$ of twice-differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with

$$f'' + f = 0.$$

Here f'' denotes the second derivative of f. The subset V is a subspace of $\mathrm{Map}(\mathbb{R}, \mathbb{R})$ (check this!).

Th vector space V contains the functions $\cos x$ and $\sin x$ since $(\cos x)' = -\sin x$ and $(\sin x)' = \cos x$.

These functions are linearly independent since if could express

$$a \cos x + b \sin x = 0 \qquad \text{for all } x \in \mathbb{R}$$

then setting $x = 0$ would imply $a = 0$ and setting $x = \pi/2$ would imply $b = 0$.

We conclude that $\dim V \geq 2$. What is $\dim V$? Is it finite? We'll answer this question in a moment.

Suppose U and V are vector spaces.

A function $f : U \to V$ is *linear* if $f(u + v) = f(u) + f(v)$ and $f(cv) = cf(v)$ for all $c \in \mathbb{R}$ and $u, v \in U$.

Define $\text{range}(f) = \{f(x) : x \in U\}$ and $\text{kernel}(f) = \{x \in U : f(x) = 0\}$.

Proposition. If $f : U \to V$ is linear then range(f) and kernel(f) are subspaces.

These subspaces are generalisations of the column space and null space of a matrix.

Proposition. If U, V, W are vector spaces and $f : V \to W$ and $g : U \to V$ are linear functions then $f \circ g : U \to V \to W$ is linear, where $f \circ g(x) = f(g(x))$.

If D is the subspace of twice-differentiable functions in $\text{Map}(\mathbb{R}, \mathbb{R})$ and $\mathcal{L} : D \to \text{Map}(\mathbb{R}, \mathbb{R})$ is the function $D(f) = f'' + f$, then $\mathcal{L}$ is a linear map and the subspace $V = \{f \in D : f'' + f = 0\}$ is our previous example is precisely kernel$(\mathcal{L})$.

To compute the dimension of this subspace, some notation is useful. Define

$0! = 1.$

$1! = 1.$

$2! = 2 \cdot 1 = 2.$

$3! = 3 \cdot 2 \cdot 1 = 6.$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24.$

$\vdots$

$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1.$

So that in general $n!$ (pronounced "n factorial") is the product of all positive integers at most n.

Now suppose we could write $f \in V$ as

$$f(x) = a_0/0! + a_1x/1! + a_2x^2/2! + a_3x^3/3! + a_4x^4/4! + \dots$$

for some real numbers $a_0, a_1, a_2, a_3, a_4, \dots \in \mathbb{R}$. Then

$$f'(x) = a_1/0! + a_2x/1! + a_3x^2/2! + a_4x^3/3! + a_5x^4/4! + \dots$$

and

$$f''(x) = a_2/0! + a_3x/1! + a_4x^2/2! + a_5x^3/3! + a_6x^4/4! + \dots$$

Since $f'' + f = 0$ we have

$$0 = (a_0 + a_2)/0! + (a_1 + a_3)x/1! + (a_2 + a_4)x^2/2! + (a_3 + a_5)x^3/3! + (a_4 + a_6)x^4/4! + \dots$$

this means

$$a_0 + a_2 = 0 \qquad \text{and} \qquad a_1 + a_3 = 0 \qquad \text{and} \qquad a_2 + a_4 = 0 \qquad \text{and} \qquad a_3 + a_5 = 0 \qquad \text{etc.}$$

Therefore $a_0 = -a_2 = a_4 = -a_6 = a_8 = \dots$ and $a_1 = -a_3 = a_5 = -a_7 = a_9 = \dots$ so

$$f(x) = a_0(1 - x^2/2! + x^4/4! - x^6/6! + \dots) + a_1(x/1! - x^3/3! + x^5/5! - x^7/7! + \dots).$$

Remembering our Taylor series from calculus, this shows that

$$f(x) = a_0 \cos x + a_1 \sin x.$$

Therefore the linearly independent functions $\cos x$ and $\sin x$ also span V, so these functions are a basis and $\dim V = 2$.