1 Last time: eigenvector and eigenvalues

Everywhere is this lecture, n denotes a positive integer.

Let A be an $n \times n$ matrix.

Definition. A vector $v \in \mathbb{R}^n$ is an *eigenvector* for A with *eigenvalue* $\lambda \in \mathbb{R}$ if $v \neq 0$ and $Av = \lambda v$.

The set of all $v \in \mathbb{R}^n$ with $Av = \lambda v$ is the λ-*eigenspace* of A for λ. This is just the nullspace of $A - \lambda I$.

Proposition. Let λ be a number. The following are equivalent:

1. There exists an eigenvector $v \in \mathbb{R}^n$ for A with eigenvalue λ.
2. The matrix $A - \lambda I$ is not invertible, where $I = I_n$ is the $n \times n$ identity matrix.
3. $\det(A - \lambda I) = 0$.
4. The λ-eigenspace for A is nonzero (that is, contains a nonzero vector).

Let x be a variable. Then $\det(A - xI)$ is a polynomial in x, called the *characteristic polynomial* of A. The eigenvalues of A are precisely the solutions to the equation

$$\det(A - xI) = 0$$

which we call the *characteristics equation* for A.

Last time we proved two nontrivial theorems:

Theorem. The eigenvalues of a triangular square matrix A are its diagonal entries. If these numbers are $d_1, d_2, \dots, d_n$ then the characteristic polynomial of A is $(d_1 - x)(d_2 - x)\cdots(d_n - x)$.

Theorem. Suppose $v_1, v_2, \dots, v_r \in \mathbb{R}^n$ are nonzero vectors. Assume each is an eigenvector for an $n \times n$ matrix A. Let λ_i be the eigenvalue corresponding to v_i, so that $Av_i = \lambda_i v_i$. If $\lambda_1, \lambda_2, \dots, \lambda_r$ are all distinct, meaning that $\lambda_i \neq \lambda_j$ if $i \neq j$, then the vectors $v_1, v_2, \dots v_r$ are linearly independent.

To illustrate these results and motivate the new topics today, we undertake a somewhat lengthy example.

Example. Consider the matrix

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Since A is triangular, its characteristic polynomial is $(1-x)(2-x)(3-x)$ and its eigenvalues are $1, 2, 3$.

1-eigenspace. The eigenvectors of A with eigenvalue 1 are the nonzero elements of $\mathrm{Nul}(A - I)$.

$$A - I = \begin{bmatrix} 0 & 5 & 4 \\ & 1 & 0 \\ & & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ & 5 & 4 \\ & & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 4 \\ & & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix} = \texttt{RREF}(A - I).$$

This shows that $x \in \mathrm{Nul}(A - I)$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis for $\mathrm{Nul}(A - I)$. Therefore all eigenvectors of A with eigenvalue 1 are nonzero scalar multiples of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

2-eigenspace. The eigenvectors of A with eigenvalue 2 are the nonzero elements of $\mathrm{Nul}(A-2I)$.

$$A - 2I = \begin{bmatrix} -1 & 5 & 4 \\ & 0 & 0 \\ & & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix} = \mathtt{RREF}(A-2I).$$

This shows that $x \in \mathrm{Nul}(A-2I)$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ is a basis for $\mathrm{Nul}(A-2I)$. All eigenvectors of A with eigenvalue 2 are nonzero scalar multiples of $\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$.

3-eigenspace. The eigenvectors of A with eigenvalue 3 are the nonzero elements of $\mathrm{Nul}(A-3I)$.

$$A - 3I = \begin{bmatrix} -2 & 5 & 4 \\ & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 4 \\ & 1 & 0 \\ & & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ & 1 & 0 \\ & & 0 \end{bmatrix} = \mathtt{RREF}(A-3I).$$

This shows that $x \in \mathrm{Nul}(A-3I)$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ so $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $\mathrm{Nul}(A-3I)$. All eigenvectors of A with eigenvalue 3 are nonzero scalar multiples of $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Since 1, 2, 3, are distinct, the second theorem implies that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent.

Consider the matrix whose columns are given by these linearly independent vectors:

$$P = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the columns of P are linearly independent, P is invertible. Recall that

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The product Pe_i is the ith column of P, so

$$Pe_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Pe_2 = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Pe_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Since $Px = y$ means that $P^{-1}y = P^{-1}Px = Ix = x$, it follows that

$$P^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1 \quad \text{and} \quad P^{-1}\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = e_2 \quad \text{and} \quad P^{-1}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = e_3.$$

Combining these identities shows that

$$P^{-1}APe_1 = P^{-1}A\begin{bmatrix}1\\0\\0\end{bmatrix} = P^{-1}\begin{bmatrix}1\\0\\0\end{bmatrix} = e_1.$$

$$P^{-1}APe_2 = P^{-1}A\begin{bmatrix}5\\1\\0\end{bmatrix} = 2P^{-1}\begin{bmatrix}5\\1\\0\end{bmatrix} = 2e_2.$$

$$P^{-1}APe_3 = P^{-1}A\begin{bmatrix}2\\0\\1\end{bmatrix} = 3P^{-1}\begin{bmatrix}2\\0\\1\end{bmatrix} = 3e_3.$$

These calculations determine the columns of the matrix $P^{-1}AP$.

If fact, we see that $P^{-1}AP = D$ where D is the diagonal matrix

$$D = \begin{bmatrix} e_1 & 2e_2 & 3e_3 \end{bmatrix} = \begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}.$$

This means that $A = P(P^{-1}AP)P^{-1} = PDP^{-1}$, i.e.,

$$\begin{bmatrix}1&5&4\\0&2&0\\0&0&3\end{bmatrix} = \begin{bmatrix}1&5&2\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1&5&2\\0&1&0\\0&0&1\end{bmatrix}^{-1}.$$

One application of this decomposition: we can derive a simple formula for an arbitrary power A^n of A.

Recall that $A^0 = I$, $A^1 = A$, $A^2 = AA$, $A^3 = AAA$, and so on.

Lemma. For any integer $n \geq 0$ we have $A^n = (PDP^{-1})^n = PD^nP^{-1}$.

Proof. Do some small examples and convince yourself that the pattern continues:

$$A^2 = AA = PDP^{-1}PDP^{-1} = PDIDP^{-1} = PD^2P^{-1}$$
$$A^3 = A^2A = PD^2P^{-1}PDP^{-1} = PD^2IDP^{-1} = PD^3P^{-1}$$
$$A^4 = A^3A = PD^3P^{-1}PDP^{-1} = PD^3IDP^{-1} = PD^4P^{-1}$$
$$\vdots$$

and so on. □

Lemma. For any integer $n \geq 0$ we have

$$D^n = \begin{bmatrix}1^n&0&0\\0&2^n&0\\0&0&3^n\end{bmatrix} = \begin{bmatrix}1&0&0\\0&2^n&0\\0&0&3^n\end{bmatrix}.$$

Proof. This is true since to multiply diagonal matrices we just multiply the entries in the corresponding diagonal positions:

$$\begin{bmatrix}x_1&&&\\&x_2&&\\&&\ddots&\\&&&x_k\end{bmatrix}\begin{bmatrix}y_1&&&\\&y_2&&\\&&\ddots&\\&&&y_k\end{bmatrix} = \begin{bmatrix}x_1y_1&&&\\&x_2y_2&&\\&&\ddots&\\&&&x_ky_k\end{bmatrix}.$$

Therefore to evaluate $D^n = DD\cdots D$, we just raise each diagonal entry to the nth power. □

Finally, by the usual algorithm we can compute $P^{-1} = \begin{bmatrix} 1 & -5 & -2 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$.

(I'm skipping the details — check that this is the correct inverse!)

Putting everything together gives the identity

$$A^n = PD^nP^{-1} = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & -5 & -2 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 5\cdot 2^n & 2\cdot 3^n \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & -5 & -2 \\ & 1 & 0 \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5(2^n-1) & 2(3^n-1) \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix}.$$

We've done all these calculations for their own sake as a means of illustrating some key concepts. But these calculations would also come up in the solution of the following discrete dynamical system. Suppose $a_0, a_1, a_2, \dots$, $b_0, b_1, b_2, \dots$, and $c_0, c_1, c_2, \dots$ are sequences of numbers. For each integer $n \geq 1$, suppose

$$a_n = a_{n-1} + 5b_{n-1} + 4c_{n-1} \qquad \text{and} \qquad b_n = 2b_{n-1} \qquad \text{and} \qquad c_n = 3c_{n-1}. \qquad (*)$$

How could we find a formula for a_n, b_n, and c_n in terms of n and the sequences' initial values a_0, b_0, c_0? Note that (*) is equivalent to

$$\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{bmatrix} = A \begin{bmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{bmatrix} = A^2 \begin{bmatrix} a_{n-2} \\ b_{n-2} \\ c_{n-2} \end{bmatrix} = \dots = A^n \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}.$$

Thus, our formula for A^n gives

$$a_n = a_0 + 5(2^n - 1)b_0 + 2(3^n - 1)c_0 \qquad \text{and} \qquad b_n = 2^n b_0 \qquad \text{and} \qquad c_n = 3^n c_0.$$

If $a_0 = b_0 = c_0 = 1$ then $a_{10} = 123212$ and $b_{10} = 1024$ and $c_{10} = 59049$. Moreover,

$$\lim_{n\to\infty} \frac{a_n}{3^n} = \lim_{n\to\infty} \frac{a_0 + 5(2^n-1)b_0 + 2(3^n-1)c_0}{3^n} = 2c_0.$$

2 Similar matrices

Definition. Two $n \times n$ matrices X and Y are *similar* if there exists an invertible $n \times n$ matrix P with $X = PYP^{-1}$. In this case observe that $Y = P^{-1}PYP^{-1}P = P^{-1}XP$. If X and Y are similar, then we say that "X is *similar to* Y" and "Y is *similar to* X." (Each statement implies the other.)

In the previous example we showed that $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ are similar matrices.

Proposition. An $n \times n$ matrix A is always similar to itself.

Proof. Since $I = I^{-1}$ we have $A = PAP^{-1}$ for $P = I$. □

Proposition. Suppose A, B, C are $n \times n$ matrices. Assume A and B are similar. Assume B and C are also similar. Then A and C are similar.

Proof. If $A = PBP^{-1}$ and $B = QCQ^{-1}$ where P, Q are invertible $n \times n$ matrices, then

$$A = PQCQ^{-1}P^{-1} = (PQ)C(PQ)^{-1} = RCR^{-1}$$

for the invertible matrix $R = PQ$. □

Theorem. If A and B are similar $n \times n$ matrices then A and B have the same characteristic polynomial and so have the same eigenvalues.

Proof. We just need to remember that $\det(XY) = \det(X)\det(Y)$ and $\det(I) = 1$.

If $A = PBP^{-1}$ then $P(A - xI)P^{-1} = PA - xP)P^{-1} = PAP^{-1} - xPP^{-1} = B - xI$.

Therefore if $A = PBP^{-1}$ then

$$\det(B - xI) = \det(P(A - xI)P^{-1}) = \det(P)\det(A - xI)\det(P^{-1}).$$

But note that $\det(P)\det(P^{-1} = \det(PP^{-1} = \det(I) = 1$, so $\det(B - xI) = \det(A - xI)$. □

Caution. Matrices may have the same eigenvalues but not be similar.

The implication goes in one direction only:

$$\text{similar} \quad \not\Leftarrow\!\!\Rightarrow \quad \text{same eigenvalues.}$$

For example, the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

both have eigenvalues $2, 2$ but are not similar.

Since $A = 2I$ we have $PAP^{-1} = 2PIP^{-1} = 2PP^{-1} = 2I = A \neq B$ for all invertible matrices P.

Caution. Row equivalence of matrices $\neq$ similarity of matrices.

Row operations usually change eigenvalues, whereas similar matrices always have the same eignenvalues.

Definition. A square matrix X is *diagonalisable* (or *diagonalizable*) if X is similar to a diagonal matrix, i.e., there exists a diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

and an invertible matrix P such that $X = PDP^{-1}$.

In our long example in the last section, we saw that $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonalisable.

Theorem. An $n \times n$ matrix A is diagonalisable if and only if the set of eigenvectors of A spans all of $\mathbb{R}^n$, or equivalently contains a subset of n linearly independent vectors.

More precisely, suppose D is an $n \times n$ diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ and P is an $n \times n$ invertible matrix with columns $v_1, v_2, \ldots, v_n$. Then $A = PDP^{-1}$ if and only if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A and $v_1, v_2, \ldots, v_n$ are eigenvectors of A such that $Av_i = \lambda_i v_i$ for $i = 1, 2, \ldots, n$.

Proof. We have

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}.$$

Then $Pe_i = v_i$ so $P^{-1}v_i =_i$ and $De_i = \lambda_i e_i$, so

$$PDP^{-1}v_i = PDe_i = \lambda_i Pe_i = \lambda_i v_i$$

for each $i = 1, 2, \dots, n$.

Therefore if $A = PDP^{-1}$ then $v_1, v_2, \dots, v_n$ are eigenvectors for A with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. In this case, as P is invertible, the columns $v_1, v_2, \dots, v_n$ must be linearly independent, so A has n linearly independent eigenvectors.

Conversely, suppose A has n linearly independent eigenvectors $v_1, v_2, \dots, v_n$ with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Define

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

as before. Since $Pe_i = v_i$ and $P^{-1}v_i = e_i$, we have

$$P^{-1}APe_i = P^{-1}Av_i = P^{-1}(\lambda_i v_i) = \lambda_i P^{-1}v_i = \lambda_i e_i.$$

This calculates the ith column of $P^{-1}AP$. Since $\lambda_i e_i$ is also the i column of the diagonal matrix D, we deduce that $P^{-1}AP = D$. Therefore $A = P(P^{-1}AP)P^{-1} = PDP^{-1}$ is diagonalisable. □

Not every matrix is diagonalisable. It takes some work to decide if a given matrix is diagonalisable. Here is one easy criterion, which is sufficient but not necessary:

Corollary. An $n \times n$ matrix with n distinct eigenvalues is diagonalisable.

Proof. Suppose A has n distinct eigenvalues. By the theorem last time, any choice of eigenvectors for A corresponding to these eigenvalues will be linearly independent, so A will have n linearly independent eigenvectors. □

Example. The matrix $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$ is triangular so has eigenvalues $5, 0, -2$.

These are distinct numbers, so A is diagonalisable.

Next time: how to "diagonalise" (that is, find P such that $A = PDP^{-1}$) a diagonalisable matrix.