1 Last time: similar and diagonalisable matrices

Let n be a positive integer. Suppose A is an $n \times n$ matrix, $v \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$.

Remember that v an *eigenvector* for A with *eigenvalue* λ if $v \neq 0$ and $Av = \lambda v$, or equivalently if v is a nonzero element of $\mathrm{Nul}(A - \lambda I)$. The number λ is an eigenvalue of A if there exists some eigenvector with this eigenvalue.

If the nullspace $\mathrm{Nul}(A - \lambda I)$ is nonzero, then it is called the λ-*eigenspace* of A.

The eigenvalues of A are the solutions to the polynomial equation $\det(A - xI) = 0$.

Important fact. Any set of eigenvectors of A with all distinct eigenvalues is linearly independent.

Two $n \times n$ matrices A and B are *similar* if there is an invertible $n \times n$ matrix P such that $A = PBP^{-1}$. Similar matrices have the same eigenvalues but usually different eigenvectors.

Example. The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$.

A matrix is *diagonal* if all of its nonzero entries appear in diagonal positions $(1,1), (2,2), \dots,$ or (n,n).

A matrix A is *diagonalisable* if it is similar to a diagonal matrix.

In other words, A is diagonalisable if we can write $A = PDP^{-1}$ where

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

is a diagonal matrix. In this case $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A are if $P = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$ then $Av_i = \lambda_i v_i$ for each $i = 1, 2, \dots, n$, i.e., the columns of P are a basis for $\mathbb{R}^n$ of eigenvectors of A.

We proved these results last time:

Theorem. An $n \times n$ matrix A is diagonalisable if and only if $\mathbb{R}^n$ has a basis whose elements are all eigenvectors of A.

Theorem. If A is an $n \times n$ matrix with n distinct eigenvalues then A is diagonalisable.

2 Diagonalisation and Fibonacci numbers

Knowing how to diagonalise matrices will let us prove an exact formula for the *Fibonacci numbers*.

The sequence f_n of Fibonacci numbers starts as

$$f_0 = 0, \quad f_1 = 1, \quad f_2 = 1, \quad f_3 = 2, \quad f_4 = 3, \quad f_5 = 5, \quad f_6 = 8, \quad f_7 = 13 \quad \dots$$

For $n \geq 2$, the sequence is defined by $f_n = f_{n-2} + f_{n-1}$.

We have $f_{10} = 55$ and $f_{100} = 354224848179261915075$.

The sequence grows exponentially. Write log for the natural logarithm. Then

$\frac{1}{10}\log f_{10} = 0.40073\ldots$
$\frac{1}{100}\log f_{100} = 0.47316\ldots$
$\frac{1}{1000}\log f_{1000} = 0.48040\ldots$
$\frac{1}{10000}\log f_{10000} = 0.481131\ldots$
$\frac{1}{1000000}\log f_{1000000} = 0.481211\ldots$

These numbers seem to be converging to something.

In fact, if we set $x = 0.481211\ldots$ and $e = 2.718\ldots$ (so that $\log e = 1$) then $(2e^x - 1)^2 = 4.999995\cdots \approx 5$.

Can we explain this?

Define $a_n = f_{2n}$ and $b_n = f_{2n+1}$ for $n \geq 0$.

If $n > 0$ then
$$a_n = f_{2n} = f_{2n-2} + f_{2n-1} = a_{n-1} + b_{n-1}.$$
Similarly, we compute that if $n > 0$ then
$$b_n = f_{2n+1} = f_{2n-1} + f_{2n} = b_{n-1} + a_n = a_{n-1} + 2b_{n-1}.$$
We can put these two equations together into one matrix equation:
$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix}.$$
Since this holds for all $n > 0$, we have
$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^2 \begin{bmatrix} a_{n-2} \\ b_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^3 \begin{bmatrix} a_{n-3} \\ b_{n-3} \end{bmatrix} = \cdots = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^n \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}.$$
In other words,
$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
Thus if we could get an exact formula for the matrix
$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^n$$
then we could derive a formula for $a_n = f_{2n}$ and $b_n = f_{2n+1}$, which would determine f_n for all n.

The best way we know to compute A^n for large values of n is to *diagonalise* A, that is, to find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$, since then $A^n = PD^nP^{-1}$. Note, however, that at the outset it's not clear if this is even possible.

Define the matrix
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$
To determine if A is diagonalisable, our first step is to compute its eigenvalues, which are the roots of the polynomial
$$\det(A - xI) = \det \begin{bmatrix} 1-x & 1 \\ 1 & 2-x \end{bmatrix} = (1-x)(2-x) - 1 = x^2 - 3x + 1.$$
By the quadratic formula, the values of x where this polynomial is zero are
$$\alpha = \frac{3+\sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{3-\sqrt{5}}{2}.$$

These numbers are the eigenvalues of A.

Since $\alpha - \beta = \sqrt{5} \neq 0$, these eigenvalues are distinct so A is diagonalisable. Note that

$$\alpha\beta = (3-\sqrt{5})(3+\sqrt{5})/4 = (9-5)/4 = 1.$$

Our next step is to find bases for the α- and β-eigenspaces of A.

To find an eigenvector for A with eigenvalue α, we row reduce

$$A-\alpha I = \left[\begin{array}{cc} 1-\alpha & 1 \\ 1 & 2-\alpha \end{array}\right] \sim \left[\begin{array}{cc} 1 & 2-\alpha \\ 1-\alpha & 1 \end{array}\right] \sim \left[\begin{array}{cc} 1 & 2-\alpha \\ 0 & 1-(2-\alpha)(1-\alpha) \end{array}\right] = \left[\begin{array}{cc} 1 & 2-\alpha \\ 0 & 0 \end{array}\right] = \texttt{RREF}(A-\alpha I).$$

The second equality holds since

$$(2-\alpha)(1-\alpha) = (1-\sqrt{5})(-1-\sqrt{5})/4 = (-1+5)/4 = 1.$$

This computation show that $x \in \mathrm{Nul}(A-\alpha I)$ if and only if $x = \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$ where $x_1 + (2-\alpha)x_2 = 0$, so

$$v = \left[\begin{array}{c} \alpha-2 \\ 1 \end{array}\right]$$

is an eigenvector for A with $Av = \alpha v$.

To find an eigenvector for A with eigenvalue β, we similarly row reduce

$$A-\beta I = \left[\begin{array}{cc} 1-\beta & 1 \\ 1 & 2-\beta \end{array}\right] \sim \left[\begin{array}{cc} 1 & 2-\beta \\ 1-\beta & 1 \end{array}\right] \sim \left[\begin{array}{cc} 1 & 2-\beta \\ 0 & 1-(2-\beta)(1-\beta) \end{array}\right] = \left[\begin{array}{cc} 1 & 2-\beta \\ 0 & 0 \end{array}\right] = \texttt{RREF}(A-\beta I).$$

The second equality holds since also $(2-\beta)(1-\beta) = 1$.

By algebra identical to the previous case, we deduce that

$$w = \left[\begin{array}{c} \beta-2 \\ 1 \end{array}\right]$$

is an eigenvector for A with $Av = \beta v$.

This means that for

$$P = \left[\begin{array}{cc} v & w \end{array}\right] = \left[\begin{array}{cc} \alpha-2 & \beta-2 \\ 1 & 1 \end{array}\right] \qquad \text{and} \qquad D = \left[\begin{array}{cc} \alpha & 0 \\ 0 & \beta \end{array}\right]$$

we have $A = PDP^{-1}$. Since P is 2×2 with $\det P = (\alpha-2)-(\beta-2) = \alpha-\beta = \sqrt{5}$, we have

$$D^n = \left[\begin{array}{cc} \alpha^n & 0 \\ 0 & \beta^n \end{array}\right] \qquad \text{and} \qquad P^{-1} = \frac{1}{\sqrt{5}}\left[\begin{array}{cc} 1 & 2-\beta \\ -1 & \alpha-2 \end{array}\right].$$

We therefore have

$$\left[\begin{array}{c} a_n \\ b_n \end{array}\right] = A^n \left[\begin{array}{c} 0 \\ 1 \end{array}\right] = PD^nP^{-1}\left[\begin{array}{c} 0 \\ 1 \end{array}\right] = \frac{1}{\sqrt{5}}\left[\begin{array}{cc} \alpha-2 & \beta-2 \\ 1 & 1 \end{array}\right]\left[\begin{array}{cc} \alpha^n & 0 \\ 0 & \beta^n \end{array}\right]\left[\begin{array}{cc} 1 & 2-\beta \\ -1 & \alpha-2 \end{array}\right]\left[\begin{array}{c} 0 \\ 1 \end{array}\right].$$

Before computing anything further, it helps to make a few simplifications. Note that

$$\alpha - 2 = \frac{-1+\sqrt{5}}{2} = 1-\beta \qquad \text{and} \qquad \beta-2 = \frac{-1-\sqrt{5}}{2} = 1-\alpha.$$

Hence

$$\begin{aligned}\begin{bmatrix} a_n \\ b_n \end{bmatrix} &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1-\beta & 1-\alpha \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} \begin{bmatrix} 1 & \alpha-1 \\ -1 & 1-\beta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1-\beta & 1-\alpha \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} \begin{bmatrix} \alpha-1 \\ 1-\beta \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1-\beta & 1-\alpha \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (\alpha-1)\alpha^n \\ -(\beta-1)\beta^n \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} (\alpha-1)(\beta-1)\,(\beta^n-\alpha^n) \\ (\alpha-1)\alpha^n - (\beta-1)\beta^n \end{bmatrix}.\end{aligned}$$

Since $(\alpha-1)(\beta-1) = \frac{(1-\sqrt{5})(1+\sqrt{5})}{4} = \frac{1-4}{4} = -1$, rewriting this matrix equation gives

(i) $f_{2n} = a_n = \frac{1}{\sqrt{5}}\,(\alpha^n - \beta^n)$.

(ii) $f_{2n+1} = b_n = \frac{1}{\sqrt{5}}\,((\alpha-1)\alpha^n - (\beta-1)\beta^n)$.

We now make one more unexpected observation:

$$(\alpha-1)^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2} = \alpha$$

and

$$(\beta-1)^2 = \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{3-\sqrt{5}}{2} = \beta.$$

Thus (i) and (ii) become

$$f_{2n} = \tfrac{1}{\sqrt{5}}\left((\alpha-1)^{2n} - (\beta-1)^{2n}\right) \tag{*}$$

and

$$f_{2n+1} = \tfrac{1}{\sqrt{5}}\left((\alpha-1)^{2n+1} - (\beta-1)(\alpha-1)^{2n+1}\right). \tag{**}$$

Putting (*) and (**) together gives a common formula for f_n for all n. Namely, since

$$\alpha - 1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta - 1 = \frac{1-\sqrt{5}}{2}$$

we get:

Theorem. For all integers $n \geq 0$ it holds that

$$\boxed{f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \approx 0.447\,(1.618^n - (-0.618)^n)}$$

(Check that this holds even when $n = 0$ and $n = 1$.)

How does this explain our original numeric observations?

Well, since $\frac{1-\sqrt{5}}{2} = -0.618\dots$ has absolute value less than 1, it follows that if n is very large then

$$f_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

so

$$\log f_n \approx \log\left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right) = \log\left(\frac{1}{\sqrt{5}}\right) + n\log\left(\frac{1+\sqrt{5}}{2}\right).$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n}\log f_n = \lim_{n\to\infty}\left(\underbrace{\frac{1}{n}\log\left(\frac{1}{\sqrt{5}}\right)}_{\to 0} + \underbrace{\frac{n}{n}}_{=1}\log\left(\frac{1+\sqrt{5}}{2}\right)\right) = \log\left(\frac{1+\sqrt{5}}{2}\right).$$

And indeed

$$\log\left(\frac{1+\sqrt{5}}{2}\right) = 0.481211\cdots \approx \frac{1}{1000000}\log f_{1000000}.$$

Moreover, if $x = \log\left(\frac{1+\sqrt{5}}{2}\right)$ then $e^x = (1+\sqrt{5})/2$ so $(2e^x-1)^2 = 5$.

3 Diagonalising matrices whose eigenvalues are not distinct

If an $n \times n$ matrix A has n distinct eigenvalues with corresponding eigenvectors $v_1, v_2, \dots, v_n$, then the matrix $P = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$ is automatically invertible since its columns are linearly independent, and the matrix $D = P^{-1}AP$ is diagonal such that $A = PDP^{-1}$.

When A is diagonalisable but has fewer than n distinct eigenvalues, we can still build up P in such a way that P is automatically invertible and $P^{-1}AP$ is automatically diagonal.

Recall that if λ is an eigenvalue of A then $\mathrm{Nul}(A - \lambda I)$ is the λ-*eigenspace* of A.

The *multiplicity* of the eigenvalue λ is the largest integer $m \geq 1$ such that we can write the characteristic polynomial of A as the product $\det(A - xI) = (\lambda - x)^m p(x)$ for some polynomial $p(x)$.

For example, if $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ then

$$\det(A - xI) = \det\begin{bmatrix} -x & -1 \\ 1 & 2-x \end{bmatrix} = (-x)(2-x) + 1 = x = x^2 - 2x + 1 = (x-1)^2$$

so 1 is an eigenvalue of A with multiplicity 2.

Theorem. Let A be an $n \times n$ matrix. Suppose A has distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_p$ where $p \leq n$. The following properties then hold:

(a) For each $i = 1, 2, \dots, p$, the dimension of the λ_i-eigenspace of A is at most the multiplicity of λ_i.

(b) A is diagonalisable if and only if the sum of the dimensions of the eigenspaces of A is n.

(c) Suppose A is diagonalisable and $\mathcal{B}_i$ is a basis for the λ_i-eigenspace. Then the union $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_p$ is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A. If the elements of this union are the vectors $v_1, v_2, \dots, v_n$ then the matrix $P = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$ is invertible and $P^{-1}AP$ is diagonal.

Proof. Fix an index $i \in \{1, 2, \dots, p\}$.

Let $\lambda = \lambda_i$ and suppose λ has multiplicity m and $\mathrm{Nul}(A - \lambda I)$ has dimension d.

Let $v_1, v_2, \dots, v_d$ be a basis for $\mathrm{Nul}(A - \lambda I)$.

One of the corollaries we saw for the dimension theorem is that it is always possible to choose vectors $v_{d+1}, v_{d+2}, \dots, v_n \in \mathbb{R}^n$ such that $v_1, v_2, \dots, v_d, v_{d+1}, v_{d+2}, \dots, v_n$ is a basis for $\mathbb{R}^n$.

Define $Q = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$. The columns of this matrix are linearly independent, so Q is invertible with $Qe_j = v_j$ and $Q^{-1}v_j = e_j$ for all $j = 1, 2, \dots, n$. Define $B = Q^{-1}AQ$.

If $j \in \{1, 2, \dots, d\}$ then the jth column of B is

$$Be_j = Q^{-1}AQe_j = Q^{-1}Av_j = \lambda Q^{-1}v_j = \lambda e_j.$$

This means that the *first d columns* of B are

$$\begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

so B has the block-triangular form

$$B = \begin{bmatrix} \lambda & & & & * & * & \dots & * \\ & \lambda & & & * & * & \dots & * \\ & & \ddots & & \vdots & \vdots & \ddots & \vdots \\ & & & \lambda & * & * & \dots & * \\ 0 & 0 & \dots & 0 & * & * & \dots & * \\ \vdots & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & * & * & \dots & * \end{bmatrix} = \left[\begin{array}{c|c} \lambda I_d & Y \\ \hline 0 & Z \end{array}\right]$$

where Y is an arbitrary $d \times (n-d)$ matrix and Z is an arbitrary $(n-d)\times(n-d)$ matrix.

From the properties of the determinant that we know, there are various ways to deduce that

$$\det(B - xI) = (\lambda - x)^d \det(Z - xI).$$

Since $\det(A - xI) = \det(B - xI)$ as A and B are similar, and since $\det(Z - xI)$ is a polynomial in x, we see that $\det(A - xI)$ can be written as $(\lambda - x)^d p(x)$ for some polynomial $p(x)$. Since m is maximal such that $\det(A - xI) = (\lambda - x)^m p(x)$, it must hold that $d \leq m$. This proves part (a).

To prove parts (b) and (c), suppose $v_i^1, v_i^2, \dots, v_i^{\ell_i}$ is a basis for the λ_i-eigenspace of A for each $i = 1, 2, \dots, p$. Let $\mathcal{B}_i = \{v_i^1, v_i^2, \dots, v_i^{\ell_i}\}$. We claim that $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \mathcal{B}_p$ is a linearly independent set.

To prove this, suppose $\sum_{i=1}^p \sum_{j=1}^{\ell_i} c_i^j v_i^j = 0$ for some coefficients $c_i^j \in \mathbb{R}$.

It suffices to show that every $c_i^j = 0$.

Let $w_i = \sum_{j=1}^{\ell_i} c_i^j v_i^j \in \mathbb{R}^n$. We then have $w_1 + w_2 + \dots + w_p = 0$.

Each w_i is either zero or an eigenvector of A with eigenvalue λ_i. (Why?)

Since eigenvectors of A with distinct eigenvalues are linearly independent, we must have

$$w_1 = w_2 = \dots = w_p = 0.$$

But since each set $\mathcal{B}_i$ is linearly independent, this implies that $c_i^j = 0$ for all i, j.

We conclude that $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \mathcal{B}_p$ is a linearly independent set.

If the sum of the dimensions of the eigenspaces of A is n then $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_p$ is a set of n linearly independent eigenvectors of A, so A is diagonalisable.

If A is diagonalisable then A has n linearly independent eigenvectors. Among these vectors, the number that can belong to any particular eigenspace of A is necessarily the dimension of that eigenspace, so it follows that sum of the dimensions of the eigenspaces of A at least n. This sum cannot be more than n since the sum is the size of the linearly independent set $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_p \subset \mathbb{R}^n$. This proves part (b).

To prove part (c), note that if A is diagonalisable then $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_p$ is a set of n linearly independent vectors in $\mathbb{R}^n$, so is a basis for $\mathbb{R}^n$. $\square$

Example. Consider the lower-triangular matrix

$$A = \begin{bmatrix} 5 & & & \\ 0 & 5 & & \\ 1 & 4 & -3 & \\ -1 & -2 & 0 & -3 \end{bmatrix}.$$

Its characteristic polynomial is $\det(A - xI) = (5-x)^2(-x-3)^2$.

The eigenvalues of A are therefore 5 and -3, each with multiplicity 2.

Since

$$A - 5I = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 1 & 4 & -8 & \\ -1 & -2 & 0 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 8 & 16 \\ 0 & 1 & -4 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \texttt{RREF}(A-5I)$$

it follows that $x \in \mathrm{Nul}(A - 5I)$ if and only if

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -8x_3 - 16x_4 \\ 4x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

so

$$\boxed{\begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix} \text{ is a basis for } \mathrm{Nul}(A-5I).}$$

Since

$$A - (-3)I = A + 3I = \begin{bmatrix} 8 & & & \\ 0 & 8 & & \\ 1 & 4 & 0 & \\ -1 & -2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \texttt{RREF}(A+3I)$$

it follows that $x \in \mathrm{Nul}(A + 3I)$ if and only if

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so

$$\boxed{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ is a basis for } \mathrm{Nul}(A+3I).}$$

Each eigenspace has dimension 2, so the sum of the dimensions of the eigenspaces of A is $2 + 2 = 4 = n$.

Thus A is diagonalisable.

In particular, if

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

then P is invertible and

$$P^{-1}AP = \begin{bmatrix} 5 & & & \\ & 5 & & \\ & & -3 & \\ & & & -3 \end{bmatrix} \qquad \text{and} \qquad A = P\begin{bmatrix} 5 & & & \\ & 5 & & \\ & & -3 & \\ & & & -3 \end{bmatrix}P^{-1}.$$