This document is intended as an **exact transcript** of the lecture, with extra summary and vocabulary sections for your convenience. By design, the material covered in lecture is exactly the same as what is in these notes. Due to time constraints, the notes may sometimes only contain limited illustrations, proofs, and examples; for a more thorough discussion of the course content, **consult the textbook**.

Summary

Quick summary of today's notes. Lecture starts on next page.

- Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation with standard matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

  If $v \in \mathbb{R}^2$, then $T(v) = Av$ is the vector in $\mathbb{R}^2$ formed by rotating v counterclockwise by θ radians.

- A function $f : X \to Y$ is *one-to-one* (or *injective*) if we never have $f(a) = f(b)$ for $a \neq b$.
- A function $f : X \to Y$ is *onto* (or *surjective*) if for each $y \in Y$, there exists $x \in X$ with $f(x) = y$.
- Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with standard matrix A.

  Then T is one-to-one if and only if the columns of A are linearly independent.

  This happens if and only if every column of A contains a pivot position.

  Likewise, T is onto if and only if the span of the columns of A is $\mathbb{R}^m$.

  This happens if and only if every row of A contains a pivot position.

1 Last time: linear transformations

A *function* consists of three things: a set of inputs called the *domain*, a rule that transforms these inputs to outputs, and a set called the *codomain* that contains all outputs (and possibly some other elements that are not outputs).

Writing $f : X \to Y$ means that X is the domain, Y is the codomain, and f is a function that transforms each input in X to an output that belongs to Y.

Let m and n be positive integers. Recall that $\mathbb{R}^n$ is the set of vectors with n rows.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a function, whose domain and codomain are the sets of vectors $\mathbb{R}^n$ and $\mathbb{R}^m$.

The following mean the same thing:

- T is *linear* is the sense that $T(u + v) = T(u) + T(v)$ and $T(cv) = cT(v)$ for $u, v \in \mathbb{R}^n$, $c \in \mathbb{R}$.
- There is an $m \times n$ matrix A such that T has the formula $T(v) = Av$ for $v \in \mathbb{R}^n$.

If we are given a linear transformation T, then $T(v) = Av$ for the matrix

$$A = \left[\begin{array}{cccc} T(e_1) & T(e_2) & \dots & T(e_n) \end{array}\right]$$

where $e_i \in \mathbb{R}^n$ is the vector with a 1 in row i and 0 in all other rows.

We call A the *standard matrix* of T.

Two different linear functions $\mathbb{R}^n \to \mathbb{R}^m$ cannot have the same standard matrix.

Example. Fix $\theta \in [0, 2\pi)$. The notation $[a, b)$ means "the set of numbers $x \in \mathbb{R}$ with $a \le x < b$." Define

$$A = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$$

and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(v) = Av$.

If $v = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$ is a vector parallel to the x-axis, then $T(v) = Av = \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array}\right]$.

If $v = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$ is a vector parallel to the y-axis, then $T(v) = Av = \left[\begin{array}{c} -\sin\theta \\ \cos\theta \end{array}\right] = \left[\begin{array}{c} \cos(\theta + \frac{\pi}{2}) \\ \sin(\theta + \frac{\pi}{2}) \end{array}\right]$.

In general, $T(v) = Av$ is the vector obtained by rotating v counterclockwise by the angle θ.

This holds since any vector $v = \left[\begin{array}{c} v_1 \\ v_2 \end{array}\right]$ can be written $v = \left[\begin{array}{c} v_1 \\ 0 \end{array}\right] + \left[\begin{array}{c} 0 \\ v_2 \end{array}\right]$, so is the arrow to the opposite vertex in the parallelogram with sides $\left[\begin{array}{c} v_1 \\ 0 \end{array}\right]$ and $\left[\begin{array}{c} 0 \\ v_2 \end{array}\right]$.

Since $T(v) = T\left(\left[\begin{array}{c} v_1 \\ 0 \end{array}\right]\right) + T\left(\left[\begin{array}{c} 0 \\ v_2 \end{array}\right]\right)$ and since T rotates by angle θ the two vectors on the right, $T(v)$ is the arrow from 0 to the opposite vertex in our previous parallelogram, now rotated by angle θ.

2 One-to-one and onto functions

This section talks about two important classes of linear transformations, which can be characterized in terms of whether the columns of the standard matrix are linearly independent or span the codomain.

Definition. A function $f : X \to Y$ is *one-to-one* (or *injective*) if $f(a) = f(b)$ implies $a = b$.

This means that f does not send two different inputs to the same output.

If $a \neq b$ and $f(a) = f(b)$ then f is not one-to-one.

Example. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear transformation $T(v) = Av$ where

$$A = \left[\begin{array}{ccc} 1 & 2 & 5 \\ 0 & 5 & 3 \end{array}\right].$$

Is T one-to-one? No: since A has more columns than rows, its columns are linearly dependent. Therefore there is a vector $0 \neq v \in \mathbb{R}^3$ such that $T(v) = Av = 0$. But we also have $T(0) = 0$.

Theorem. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then the following mean the same thing:

(a) T is one-to-one.

(b) The only solution to $T(x) = 0$ is $x = 0 \in \mathbb{R}^n$.

(c) The columns of the standard matrix A of T are linearly independent.

Proof. Suppose the only solution to $T(x) = 0$ is $x = 0 \in \mathbb{R}^n$. Then whenever $u, v \in \mathbb{R}^n$ are vectors with $u \neq v$, we have $T(u) - T(v) = T(u - v) \neq 0$ since $u - v \neq 0$, so $T(u) \neq T(v)$. Therefore T is one-to-one.

If T is one-to-one, then $T(x) = T(0) = 0$ implies $x = 0$, so the only solution to $T(x) = 0$ is $x = 0$. □

Definition. A function $f : X \to Y$ is *onto* (or *surjective*) if $\mathsf{range}(f) = \{f(x) : x \in X\} = Y$.

Thus, f is onto if its range is equal to its codomain.

If there is a value $y \in Y$ such that $f(x) \neq y$ for all $x \in X$, then f is not onto.

Example. Suppose again that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear transformation $T(v) = Av$ where

$$A = \left[\begin{array}{ccc} 1 & 2 & 5 \\ 0 & 5 & 3 \end{array}\right].$$

Is T onto? Yes: the span of the columns of A is $\mathbb{R}^2$ if and only if A has a pivot in every row, and

$$A = \left[\begin{array}{ccc} 1 & 2 & 5 \\ 0 & 5 & 3 \end{array}\right] \sim \left[\begin{array}{ccc} 1 & 2 & 5 \\ 0 & 1 & 3/5 \end{array}\right] \sim \left[\begin{array}{ccc} 1 & 0 & 19/5 \\ 0 & 1 & 3/5 \end{array}\right] = \mathsf{RREF}(A).$$

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation with standard matrix A.

The following properties are all equivalent:

(a) T is onto.

(b) The matrix equation $Ax = b$ has a solution for each $b \in \mathbb{R}^m$

(c) The span of the columns of A is $\mathbb{R}^m$.

Proof. The vectors in the range of T are precisely the linear combinations of the columns of A.

This is $\mathbb{R}^m$ precisely when the span of the columns of A is $\mathbb{R}^m$. □

Example. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function $T\left(\left[\begin{array}{c} v_1 \\ v_2 \end{array}\right]\right) = \left[\begin{array}{r} 3v_1 + v_2 \\ 5v_1 + 7v_2 \\ v_1 + 3v_2 \end{array}\right].$

This function is a linear transformation. Its standard matrix is $A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$.

To determine if T is one-to-one, we check if the columns of A are linearly independent. To do this, we convert A to its reduced echelon form:

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & -8 \\ 0 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \mathsf{RREF}(A).$$

This shows that A has a pivot position in every column, which means that the only solution to $Ax = 0$ is $x = 0$, which means the columns of A are linearly independent, which means T is one-to-one.

To determine if T is onto, we want to find out if the columns of A span $\mathbb{R}^3$. From last time, we know that this happens if and only if A has a pivot position in every row. Since the third row of A has no pivot position, T is not onto.

Corollary. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one only if $n \leq m$, and onto only if $n \geq m$.

Proof. Let A be the standard matrix of T.

Results last time show that T is one-to-one if and only if A has a pivot position in every column, and T is onto if and only if A has a pivot position in every row.

Each row and each column contains at most one pivot position. Thus if A has a pivot in every column then the number of columns n cannot be more than the number of rows m. Likewise, if A has a pivot in every row then the number of rows m cannot be more than the number of columns n.

This means that if T is one-to-one then $n \leq m$ and if T is onto then $m \leq n$. □

There is a more efficient way to state the conditions that characterize when a linear transformation is one-to-one or onto:

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with standard matrix A.

(a) The linear transformation T is one-to-one if and only if the matrix A has a pivot in every column.

(b) The linear transformation T is onto if and only if the matrix A has a pivot in every row.

Proof. We have seen that T is one-to-one if and only if the columns of A are linearly independent, which occurs precisely when A has a pivot in every column.

Likewise, we have seen that T is onto if and only if the span of the columns of A is $\mathbb{R}^m$, which occurs precisely when A has a pivot in every row. □

3 Vocabulary

Keywords from today's lecture:

1. **One-to-one** or **injective** function $f : X \to Y$.

   A function with the property that if $f(u) = f(v)$ for $u, v \in X$ then $u = v$.

   Example: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$.

   The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not one-to-one: $f(-2) = f(2) = 4$.

2. **Onto** or **surjective** function $f : X \to Y$.

   A function with the property that $y \in Y$ then there exists $x \in X$ with $f(x) = y$.

   Example: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$.

   The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not onto: no negative number is in its range.