This document is intended as an **exact transcript** of the lecture, with extra summary and vocabulary sections for your convenience. By design, the material covered in lecture is exactly the same as what is in these notes. Due to time constraints, the notes may sometimes only contain limited illustrations, proofs, and examples; for a more thorough discussion of the course content, **consult the textbook**.

Summary

Quick summary of today's notes. Lecture starts on next page.

- Let n be a positive integer and let A and B be $n \times n$ matrices.
- It always holds that $\det A = \det A^T$.
- If A is invertible then $\det A \neq 0$. If A is not invertible then $\det A = 0$.
- It always holds that $\det AB = (\det A)(\det B)$.
- A matrix is *triangular* if it looks like

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$$

  where the $*$'s are arbitrary entries.

  Let $a_{ij} \in \mathbb{R}$ denote the entry of A in the ith row and jth column.

  If A is triangular then $\boxed{\det A = a_{11}a_{22}a_{33}\cdots a_{nn}}$ = the product of the diagonal entries of A.

  The matrix A is *diagonal* if $a_{ij} = 0$ whenever $i \neq j$. Diagonal matrices are triangular.
- Here is an algorithm to compute $\det A$:
  - Perform a series of row operations to transform A to a matrix E in echelon form.
  - Keep track of a scalar $\mathsf{denom} \in \mathbb{R}$ as you do this. Start with $\mathsf{denom} = 1$.
  - Whenever you swap two rows of A, multiply denom by -1.
  - Whenever you multiply a row of A by a nonzero number, multiply denom by that number.
  - Then $\boxed{\det A = \dfrac{\det E}{\mathsf{denom}} = \dfrac{\text{product of diagonal entries of } E}{\mathsf{denom}}}$.
- Here is another way to compute $\det A$.

  Again write a_{ij} for the entry of A in row i and column j.

  Also let $A^{(i,j)}$ be the matrix formed from A by deleting row i and column j.

  Then $\boxed{\det A = a_{11} \det A^{(1,1)} - a_{12} \det A^{(1,2)} + a_{13} \det A^{(1,3)} - \cdots - (-1)^n a_{1n} \det A^{(1,n)}}$.

  This formula is complicated and inefficient for generic matrices.

  It is useful when many entries of A are equal to zero, since then the formula has few terms.

  Also, when $n \leq 3$ and you expand all the terms in this formula, you get the identities

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \quad \text{and} \quad \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg).$$

1 Last time: introduction to determinants

Let n be a positive integer.

A *permutation matrix* is a square matrix formed by rearranging the columns of the identity matrix.

Equivalently, a permutation matrix is a square matrix whose entries are all 0 or 1, and that has exactly one nonzero entry in each row and in each column.

Let S_n be the set of $n \times n$ permutation matrices.

If A is an $n \times n$ matrix and $X \in S_n$, then AX has the same columns as A but in a different order.

The columns of A are "permuted" by X to form AX.

Example. The six elements of S_3 are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Given $X \in S_n$ and an arbitrary $n \times n$ matrix A:

- Define $\mathsf{prod}(X, A)$ to be the product of the entries of A in the nonzero positions of X.
- Define $\mathsf{inv}(X)$ to be the number of 2×2 submatrices of X equal to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

To form a 2×2 submatrix of X, choose any two rows and any two columns, not necessarily adjacent, and then take the 4 entries determined by those rows and columns.

Each 2×2 submatrix of a permutation matrix is either

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example. $\mathsf{prod}\left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) = cdh$

Example. $\mathsf{inv}\left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right) = 2$ and $\mathsf{inv}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$ and $\mathsf{inv}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) = 3.$

Definition. The *determinant* of an $n \times n$ matrix A is the number given by the formula

$$\boxed{\det A = \sum_{X \in S_n} \mathsf{prod}(X, A)(-1)^{\mathsf{inv}(X)}}$$

This general formula simplifies to the following expressions for $n = 1, 2, 3$:

$$\det \begin{bmatrix} a \end{bmatrix} = a.$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - ef).$$

For $n \geq 4$, our formula for $\det A$ is a sum with at least 24 terms, which is not easy to compute by hand (or with a computer, for slightly larger n). We will describe a better way to compute determinants today.

The most important properties of the determinant are described by the following theorem:

Theorem. The determinant is the unique function $\det : \{n \times n \text{ matrices}\} \to \mathbb{R}$ with these 3 properties:

(1) $\boxed{\det I_n = 1}$.

(2) If B is formed by switching two columns in an $n \times n$ matrix A, then $\boxed{\det A = -\det B}$.

(3) Suppose A, B, and C are $n \times n$ matrices with columns

$$A = \left[\begin{array}{cccc} a_1 & a_2 & \dots & a_n \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{cccc} b_1 & b_2 & \dots & b_n \end{array}\right] \quad \text{and} \quad C = \left[\begin{array}{cccc} c_1 & c_2 & \dots & c_n \end{array}\right].$$

Assume there is an index i where $a_i = pb_i + qc_i$ for numbers $p, q \in \mathbb{R}$.

Assume also that $a_j = b_j = c_j$ for all other indices $j \in \{1, 2, \dots, i-1, i+1, i+2, \dots, n\}$.

Then $\boxed{\det A = p \det B + q \det C}$.

Corollary. If A is a square matrix that is not invertible then $\det A = 0$.

Corollary. If A is a permutation matrix then $\det A = (-1)^{\mathsf{inv}(A)}$.

Proof. $\mathsf{prod}(X, Y) = 0$ if X and Y are different $n \times n$ permutation matrices, but $\mathsf{prod}(X, X) = 1$. $\square$

2 More properties of the determinant

Recall that A^T denotes the transpose of a matrix A (the matrix whose rows are the columns of A).

Lemma. If $X \in S_n$ then $X^T \in S_n$ and $\mathsf{inv}(X) = \mathsf{inv}(X^T)$.

Proof. Transposing a permutation matrix does not affect the # of 2×2 submatrices equal to $\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$. $\square$

Corollary. If A is any square matrix then $\det A = \det(A^T)$.

Proof. If $X \in S_n$ then $\mathsf{prod}(X, A) = \mathsf{prod}(X^T, A^T)$, so our formula for the determinant gives

$$\det A = \sum_{X \in S_n} \mathsf{prod}(X, A)(-1)^{\mathsf{inv}(X)} = \sum_{X \in S_n} \mathsf{prod}(X^T, A^T)(-1)^{\mathsf{inv}(X^T)}.$$

As X ranges over all elements of S_n, the transpose X^T also ranges over all elements of S_n.
The second sum is therefore equal to $\sum_{X \in S_n} \mathsf{prod}(X, A^T)(-1)^{\mathsf{inv}(X)} = \det(A^T)$. $\square$

Corollary. If A is a square matrix with two equal rows then $\det A = 0$.

Proof. In this case A^T has two equal columns, so $0 = \det A^T = \det A$. $\square$

The following lemma is a weaker form of a statement we will prove later in the lecture.

Lemma. Let A and B be $n \times n$ matrices with $\det A \neq 0$. Then $\det(AB) = (\det A)(\det B)$.

Proof. Define $f : \{\ n \times n \text{ matrices }\} \to \mathbb{R}$ to be the function $f(M) = \frac{\det(AM)}{\det A}$.

Then f has the defining properties of the determinant, so must be equal to det since det is the unique function with these properties. In more detail:

- We have $f(I_n) = \frac{\det(AI_n)}{\det A} = \frac{\det A}{\det A} = 1$.
- If M' is given by swapping two columns in M, then AM' is given by swapping the two corresponding columns in AM, so $f(M') = \frac{\det(AM')}{\det A} = \frac{-\det(AM)}{\det A} = -f(M)$.
- If column i of M is p times column i of M' plus q times column i of M'' and all other columns of M, M', and M'' are equal, then the same is true of AM, AM', and AM'' so

$$f(M) = \frac{\det(AM)}{\det A} = \frac{p\det(AM') + q\det(AM'')}{\det A} = pf(M') + qf(M'').$$

These properties uniquely characterize det, so f and det must be the same function.

Therefore $f(B) = \frac{\det(AB)}{\det A} = \det B$ for any $n \times n$ matrix B, so $\det(AB) = (\det A)(\det B)$. $\square$

3 Determinants of triangular and invertible matrices

An $n \times n$ matrix A is *upper-triangular* if all of its nonzero entries occur in positions on or above the diagonal positions $(1,1), (2,2), (3,3), \dots, (n,n)$. Such a matrix looks like

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

where the $*$ entries can be any numbers. The zero matrix is considered to be upper-triangular.

An $n \times n$ matrix A is *lower-triangular* if all of its nonzero entries occur in positions on or below the diagonal positions. Such a matrix looks like

$$\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$$

where the $*$ entries can again be any numbers. The zero matrix is also considered to be lower-triangular.

The transpose of an upper-triangular matrix is lower-triangular, and vice versa.

We say that a matrix is *triangular* if it is either upper- or lower-triangular.

A matrix is *diagonal* if it is both upper- and lower-triangular.

This happens precisely when all nonzero entries are on the diagonal: $\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$

The *diagonal entries* of A are the numbers that occur in positions $(1,1), (2,2), (3,3), \dots, (n,n)$.

Proposition. If A is a triangular matrix then $\det A$ is the product of the diagonal entries of A.

For example, we have $\det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = abc$.

Proof. Assume A is upper-triangular. If $X \in S_n$ and $X \neq I_n$ then at least one nonzero entry of X is in a position below the diagonal, in which case $\mathsf{prod}(X, A)$ is a product of numbers which includes 0 (since all positions below the diagonal in A contain zeros) and is therefore 0.

Hence $\det A = \sum_{X \in S_n} \mathsf{prod}(X, A)(-1)^{\mathsf{inv}(X)} = \mathsf{prod}(I_n, A) =$ the product of the diagonal entries of A.

If A is lower-triangular then the same result follows since $\det A = \det(A^T)$. □

Lemma. If A is an $n \times n$ matrix then $\det A$ is a nonzero multiple of $\det(\mathsf{RREF}(A))$.

Proof. Suppose B is obtained from A by an elementary row operation. To prove the lemma, it is enough to show that $\det B$ is a nonzero multiple of $\det A$. There are three possibilities for B:

1. If B is formed by swapping two rows of A then $B = XA$ for a permutation matrix $X \in S_n$.

   Therefore $\det B = \det(XA) = (\det X)(\det A) = \pm \det A$.

2. Suppose B is formed by rescaling a row of A by a nonzero scalar $\lambda \in \mathbb{R}$.

   Then $B = DA$ where D is a diagonal matrix of the form

$$D = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \lambda & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

   and in this case $\det D = \lambda \neq 0$, so $\det B = \det(DA) = (\det D)(\det A) = \lambda \det A$.

3. Suppose B is formed by adding a multiple of row i of A to row j.

   Then $B = TA$ for a triangular matrix T whose diagonal entries are all 1 and whose only other nonzero entry appears in column i and row j.

   For example, if $n = 4$ and B is formed by adding 5 times row 2 of A to row 3 then

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A.$$

   We therefore have $\det B = \det(TA) = (\det T)(\det A) = \det A$.

This shows that performing any elementary row operation to A multiplies $\det A$ by a nonzero number. It follows that $\det(\mathsf{RREF}(A))$ is a sequence of nonzero numbers times $\det A$. □

This brings us to an important property of the determinant that is worth remembering.

Theorem. An $n \times n$ matrix A is an invertible if and only if $\det A \neq 0$.

Proof. We have already seen that if A is not invertible then $\det A = 0$.

Assume A is invertible. Then $\mathsf{RREF}(A) = I_n$, so $\det(\mathsf{RREF}(A)) = \det I_n = 1$.

Hence $\det A \neq 0$ since $\det A$ is a nonzero multiple of $\det(\mathsf{RREF}(A))$. □

Our next goal is to show that the determinant is a *multiplicative function*.

Lemma. Let A and B be $n \times n$ matrices. If A or B is not invertible then AB is not invertible.

Proof. Let X and Y be $n \times n$ matrices.

We have seen that X and Y are inverses of each other if $XY = I_n$, in which case also $YX = I_n$.

Suppose AB is invertible with inverse X. Then $(AB)X = X(AB) = I_n$.

Then A is invertible with $A^{-1} = BX$ since $A(BX) = (AB)X = I_n$.

Likewise, B is invertible with $B^{-1} = XA$ since $(XA)B = X(AB) = I_n$.

Thus, if A or B is not invertible then AB cannot be invertible. $\square$

Theorem. If A and B are any $n \times n$ matrices then $\det(AB) = (\det A)(\det B)$.

Proof. We already proved this in the case when $\det A \neq 0$.

If $\det A = 0$, then A is not invertible, so AB is not invertible either, so $\det(AB) = 0 = (\det A)(\det B)$. $\square$

It is difficult to derive this theorem directly from the formula $\det A = \sum_{X \in S_n} \mathsf{prod}(X, A)(-1)^{\mathsf{inv}(X)}$.

Example. We have $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4 - 6 = -2$ and $\det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = 10 - 12 = -2$.

On the other hand, $\det \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \right) = \det \begin{bmatrix} 10 & 13 \\ 22 & 29 \end{bmatrix} = 290 - 286 = 4$.

4 Computing determinants

Our proof that $\det A$ is a nonzero multiple of $\det(\mathsf{RREF}(A))$ can be turned into an effective algorithm.

<u>Algorithm to compute $\det A$ (useful when A is larger than 3×3).</u>

Input: an $n \times n$ matrix A.

1. Start by setting a scalar $\mathsf{denom} = 1$.
2. Row reduce A to an echelon form E. It is not necessary to bring A all the way to *reduced* echelon form. We just need to row reduce A until we get an upper triangular matrix.

   Each time you perform a row operation in this process, modify denom as follows:

   (a) When you switch two rows, multiply denom by -1.

   (b) When you multiply a row by a nonzero scalar λ, multiply denom by λ.

   (c) When you add a multiple of a row to another row, don't do anything to denom.

The determinant $\det E$ is the product of the diagonal entries of E

The determinant of A is given by $\det A = \dfrac{\det E}{\mathsf{denom}}$.

Example. We reduce the following matrix to echelon form:

$$\begin{aligned}
A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -3 & -9 \\ 2 & 4 & 6 \end{bmatrix} & & \mathsf{denom} = 1 \\
\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -3 & -9 \\ 0 & -2 & -4 \end{bmatrix} & \quad \text{(we added a multiple of row 1 to row 3)} & \mathsf{denom} = 1 \\
\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & -2 & -4 \end{bmatrix} & \quad \text{(we multiplied row 2 by } -1/3\text{)} & \mathsf{denom} = -1/3 \\
\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} = E & \quad \text{(we added a multiple of row 2 to row 3)} & \mathsf{denom} = -1/3
\end{aligned}$$

Therefore $\det A = \frac{\det E}{\mathsf{denom}} = \frac{1 \cdot 1 \cdot 2}{-1/3} = -6$.

<u>Another algorithm to compute $\det A$ (useful when A has many entries equal to zero).</u>

Define $A^{(i,j)}$ to be the submatrix formed by removing row i and column j from A.

For example, if $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then $A^{(1,2)} = \begin{bmatrix} d & f \\ g & i \end{bmatrix}$.

Theorem. If A is the $n \times n$ matrix with entry a_{ij} row i and column j, then

(1) $\boxed{\det A = a_{11} \det A^{(1,1)} - a_{12} \det A^{(1,2)} + a_{13} \det A^{(1,3)} - \cdots - (-1)^n a_{1n} \det A^{(1,n)}}$.

(2) $\boxed{\det A = a_{11} \det A^{(1,1)} - a_{21} \det A^{(2,1)} + a_{31} \det A^{(3,1)} - \cdots - (-1)^n a_{n1} \det A^{(n,1)}.}$

Note that each $A^{(1,j)}$ or $A^{(j,1)}$ is a square matrix smaller than A.

Thus $\det A^{(1,j)}$ or $\det A^{(j,1)}$ can be computed by the same formula on a smaller scale.

Proof. The second formula follows from the first formula since $\det A = \det(A^T)$. (Why?)

The first formula is a consequence of the formula for $\det A$ we derived last lecture. One needs to show

$$-(-1)^j a_{1j} \det A^{(1,j)} = \sum_{X \in S_n^{(j)}} \mathsf{prod}(X, A)(-1)^{\mathsf{inv}(X)}$$

where $S_n^{(j)}$ is the set of $n \times n$ permutation matrices which have a 1 in column j of the first row.

Summing the left expression over $j = 1, 2, \ldots, n$ gives the desired formula.

Summing the right expression over $j = 1, 2, \ldots, n$ gives $\sum_{X \in S_n} \mathsf{prod}(X, A)(-1)^{\mathsf{inv}(X)} = \det A$. □

Example. This result can be used to derive our formula for the determinant of a 3-by-3 matrix:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg).$$

5 Vocabulary

Keywords from today's lecture:

1. **Upper-triangular matrix**.

   A square matrix of the form $\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$ with zeros in all positions below the main diagonal.

2. **Lower-triangular matrix**.

   A square matrix of the form $\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$ with zeros in all positions above the main diagonal.

   The transpose of an upper-triangular matrix.

3. **Triangular matrix**.

   A matrix that is either upper-triangular or lower-triangular.

4. **Diagonal matrix**.

   A square matrix of the form $\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$ with zeros in all non-diagonal positions.

   A matrix that is both upper-triangular and lower-triangular.