**Instructions:** Choose **2 problems** and write down detailed solutions, showing all necessary work. You can earn up to **8 extra credit points** by correctly solving additional problems.[1]

Feel free to discuss problems with other students but write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit.

To get full credit for the offline homework, you just need to make a good-faith attempt on two problems. The bar for receiving extra credit points is higher: your solutions need to be close to completely correct.

Show all steps and provide justification for all answers.

1. Find an orthogonal matrix $U$ and a diagonal matrix $D$ such that

$$A = UDU^T \quad \text{for the matrix } A = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 8 & 8 \\ 4 & 8 & 8 \end{bmatrix}.$$

2. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

Show all steps in your derivation by hand to receive credit (but feel free to check your answer with a calculator).

3. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

Show all steps in your derivation by hand to receive credit (but feel free to check your answer with a calculator).

4. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

Show all steps in your derivation by hand to receive credit (but feel free to check your answer with a calculator).

5. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Show all steps in your derivation by hand to receive credit (but feel free to check your answer with a calculator).

[1] There will be ∼11 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼88 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

6. Suppose $A$ is a $2 \times 2$ matrix with a singular value decomposition

$$A = U\Sigma V^T$$

where $U$ and $V$ are orthogonal $2 \times 2$ matrices and

$$\Sigma = \left[\begin{array}{rr} 10 & 0 \\ 0 & 5 \end{array}\right].$$

The first column of $U$ is the vector $\left[\begin{array}{r} -4/5 \\ 3/5 \end{array}\right]$.

Draw a picture of the region in $\mathbb{R}^2$ given by

$$\left\{ Ax : x = \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] \in \mathbb{R}^2 \text{ is a vector with } x_1^2 + x_2^2 \leq 1 \right\}.$$

7. Suppose $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ are orthonormal vectors. Assume $k < n$.

   (a) Describe an algorithm to find $n - k$ vectors

   $$v_{k+1}, v_{k+2}, \dots, v_n \in \mathbb{R}^n$$

   such that $v_1, v_2, \dots, v_n$ is an orthonormal basis for $\mathbb{R}^n$.

   (b) Suppose $n = 3$ and $v_1 = \left[\begin{array}{r} 1/3 \\ -2/3 \\ 2/3 \end{array}\right]$.

   Find two vectors $v_2, v_3 \in \mathbb{R}^3$ such that $v_1, v_2, v_3$ is an orthonormal basis for $\mathbb{R}^n$.

*8. Suppose $A$ is an $m \times n$ matrix with at most one nonzero entry in each row and column.

Describe a singular value decomposition for $A$.

Use the following notation in your answer: suppose the positions of $A$ with nonzero entries are

$$(i_1, j_1), (i_2, j_2), \dots, (i_r, j_r)$$

where $j_1 < j_2 < \dots < j_r$, and the entries in these positions are $a_1, a_2, \dots, a_r \in \mathbb{R}$.

(It may be useful to compare your answer with #2 and #5.)

*9. Suppose $A$ is an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$.

Suppose $\sigma_k$ is much larger than $\sigma_{k+1}$ where $k < \mathrm{rank}(A)$.

Describe an algorithm using an SVD for $A$ that produces a good rank $k$ approximation to $A$.

Apply this algorithm (using a computer or calculator) to find a rank 2 approximation to the matrix

$$A = \left[\begin{array}{rrr} 0.2 & 0.1 & -0.1 \\ 1.2 & 0.1 & 0.8 \\ 1.0 & -2.0 & 5.5 \end{array}\right].$$

*10. Suppose $A$ is an invertible $3 \times 3$ matrix with a singular value decomposition

$$A = \left[\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\right] \left[\begin{array}{ccc} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{array}\right] \left[\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}\right]^\top.$$

Give a geometric interpretation of the numbers $\sigma_1, \sigma_2, \sigma_3$ and the vectors $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}^3$.

Your answer should involve the sphere $\mathbb{S} = \{w \in \mathbb{R}^3 : \|w\| \leq 1\}$ and its image $A\mathbb{S} = \{Aw : w \in \mathbb{S}\}$.