SOLUTIONS TO FINAL EXAMINATION – MATH 2121, FALL 2022

Problem 1. (20 points) This question has five parts. Each part asks you to provide a definition and then give a short derivation of a related property.

(a) Give the definition of a *linear function* $T : V \to W$ from a vector space V to a vector space W.

Then explain why your definition implies $T(0) = 0$ if $T : V \to W$ is linear.

Solution:

T is linear if $T(x + y) = T(x) + T(y)$ and $T(cx) = cT(x)$ for all $x, y \in V$ nd $c \in \mathbb{R}$.

If T is linear then $T(0) = T(0 + 0) = T(0) + T(0)$ so $T(0) = 0$.

(b) Give the definition of the *span* of three vectors $u, v, w \in V$ in a vector space.

Then explain why your definition implies that u, v, and w are each contained in $\mathbb{R}\text{-span}\{u, v, w\}$.

Solution:

The span of the vectors is the set of all vectors of the form $au+bv+cw$ for $a, b, c \in \mathbb{R}$.

Each individual vector is in the span because

$$u = 1u + 0v + 0w, \quad v = 0u + 1v + 0w, \quad \text{and} \quad w = 0u + 0v + 1w.$$

(c) Define what it means for three vectors $u, v, w \in V$ in a vector space to be *linearly dependent*.

Then explain why your definition implies that u, v, w are linearly dependent if $v = 0$.

Solution:

The vectors are linearly dependent if $au+bv+cw = 0$ for some numbers $a, b, c \in \mathbb{R}$ with $a \neq 0$ or $b \neq 0$ or $c \neq 0$.

If $v = 0$ then $0u + 1v + 0w = v = 0$ so the vectors are linearly dependent.

(d) Define what it means for a subset H to be a *subspace* of a vector space V.

Then explain why your definition implies that the null space
$$\mathsf{Nul}(A) = \{v \in \mathbb{R}^n : Av = 0\}$$
of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Solution:

H is a subspace if $0 \in H$, and $x + y \in H$ and $cx \in H$ for all $x, y \in H$ and $c \in \mathbb{R}$.

Since $A0 = 0$ we have $0 \in \mathsf{Nul}(A)$.

If $x, y \in \mathsf{Nul}(A)$ then $A(x + y) = Ax + Ay = 0 + 0 = 0$ so $x + y \in \mathsf{Nul}(A)$.

If $c \in \mathbb{R}$ and $x \in \mathsf{Nul}(A)$ then $A(cx) = c(Ax) = c0 = 0$ so $cx \in \mathsf{Nul}(A)$.

Therefore $\mathsf{Nul}(A)$ is a subspace.

(e) Define what it means for a set of vectors to be a *basis* of a vector space V.

Then explain why your definition implies that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^3$.

Solution:

A set of vectors is a basis for a subspace if the vectors are linearly independent and every element of the subspace is a linear combination of the vectors.

The vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent since if
$$a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$
then
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$
so we must have $a = b = c = 0$.

Similarly, any vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ is the linear combination
$$a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
so the vectors are a basis for $\mathbb{R}^3$.

Problem 2. (10 points) Let $A = \begin{bmatrix} -1 & 3 & -9 \\ 2 & 4 & -2 \\ 3 & 2 & 5 \end{bmatrix}$.

Compute the **reduced echelon form** $\mathsf{RREF}(A)$ of A.

Then find a **basis for** $\mathsf{Col}(A)$ and a **basis for** $\mathsf{Nul}(A)$. What is the **rank** of A?

Show all steps in your calculations to receive full credit.

Solution:

We start by row reducing

$$A = \begin{bmatrix} -1 & 3 & -9 \\ 2 & 4 & -2 \\ 3 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} -1 & 3 & -9 \\ 0 & 10 & -20 \\ 3 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} -1 & 3 & -9 \\ 0 & 10 & -20 \\ 0 & 11 & -22 \end{bmatrix}$$

$$\to \begin{bmatrix} -1 & 3 & -9 \\ 0 & 1 & -2 \\ 0 & 11 & -22 \end{bmatrix} \to \begin{bmatrix} -1 & 3 & -9 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A).$$

The pivot positions are in the first two columns, so these columns of A, namely

$$\boxed{\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}}$$

are a basis for $\mathsf{Col}(A)$. Therefore $\boxed{\operatorname{rank}(A) = 2}$.

We also deduce that $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ if and only if

$$0 = \mathsf{RREF}(A) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_3 \\ x_2 - 2x_3 \\ 0 \end{bmatrix}$$

which means that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

Thus

$$\boxed{\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}}$$

is a basis for $\mathsf{Nul}(A)$.

Problem 3. (20 points) This problem has four parts.

Consider the $2n \times 2n$ matrix

$$A = \begin{bmatrix} a & & & & & & & b \\ & a & & & & & b & \\ & & \ddots & & & \iddots & & \\ & & & a & b & & & \\ & & & b & a & & & \\ & & \iddots & & & \ddots & & \\ & b & & & & & a & \\ b & & & & & & & a \end{bmatrix}$$

that has $a \in \mathbb{R}$ in all positions on the main diagonal, $b \in \mathbb{R}$ in all positions on the main anti-diagonal, and zeros in all other positions. For example, if $n = 3$ then

$$A = \begin{bmatrix} a & 0 & 0 & 0 & 0 & b \\ 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & b & a & 0 & 0 \\ 0 & b & 0 & 0 & a & 0 \\ b & 0 & 0 & 0 & 0 & a \end{bmatrix}.$$

However, for this problem we consider n to be an unspecified positive integer.

(a) Compute the **determinant** of A.

Show all steps in your calculations to receive full credit.

Solution to part (a):

If $a = 0$ then A becomes the diagonal matrix bI after swapping columns 1 and $2n$, columns 2 and $2n-1$, $\dots$, and columns n and $n+1$ (n swaps in total). Each swap multiples the determinant's value by -1, so in this case

$$\det(A) = (-1)^n \det(bI) = (-1)^n b^{2n} = (-b^2)^n.$$

If $a \neq 0$ then by adding $-\frac{b}{a}$ times row i to row $2n+1-i$ for $i = 1, 2, \dots, n$, we can transform A to a row equivalent triangular matrix B whose diagonal entries are $a, a, \dots, a$ (n times) then $a - b^2/a$, $a - b^2/a$, $\dots$, $a - b^2/a$ (also n times). None of these row operations change the value of the determinant, so

$$\det(A) = \det(B) = a^n(a - b^2/a)^n = (a^n - b^2)^n.$$

In both cases $\boxed{\det(A) = (a^2 - b^2)^n}$.

(b) Find all **eigenvalues** of A.

Solution to part (b):

The eigenvalues of A are the roots of the characteristic polynomial $\det(A - xI)$. But $A - xI$ has the same form as A just with a replaced by $a - x$, so

$$\det(A - xI) = ((a-x)^2 - b^2)^n.$$

This polynomial factors as

$$\det(A-xI) = ((a-x)^2 - b^2)^n = ((a-b-x)(a+b-x))^n = (a-b-x)^n(a+b-x)^n$$

so the only eigenvalues are $\boxed{\lambda = a - b \text{ and } \lambda = a + b}$.

(c) Find an orthogonal matrix U and a diagonal matrix D such that

$$A = UDU^\top = UDU^{-1}.$$

(Remember that an *orthogonal matrix* has orthonormal columns).

Solution to part (c):

The vectors $e_i + e_{2n+1-i}$ for $i = 1, 2, 3, \dots, n$ are eigenvectors with eigenvalue $a+b$.

The vectors $e_i - e_{2n+1-i}$ for $i = n, \dots, 3, 2, 1$ are eigenvectors with eigenvalue $a-b$.

These vectors are orthogonal and all of length $\sqrt{2}$. So one choice for U is

$$\boxed{U = \frac{1}{\sqrt{2}} \begin{bmatrix} +1 & & & & & & & +1 \\ & +1 & & & & & +1 & \\ & & \ddots & & & \iddots & & \\ & & & +1 & +1 & & & \\ & & & +1 & -1 & & & \\ & & \iddots & & & \ddots & & \\ & +1 & & & & & -1 & \\ +1 & & & & & & & -1 \end{bmatrix}}$$

and the corresponding value of D is

$$\boxed{D = \begin{bmatrix} a+b & & & & & & & \\ & a+b & & & & & & \\ & & \ddots & & & & & \\ & & & a+b & & & & \\ & & & & a-b & & & \\ & & & & & \ddots & & \\ & & & & & & a-b & \\ & & & & & & & a-b \end{bmatrix}}.$$

(d) Find all values of $a, b \in \mathbb{R}$ such that A is invertible and compute A^{-1}.

Solution to part (d):

The matrix A is invertible if and only if its determinant is nonzero, which happens when $a^2 \neq b^2$ or equivalently when $|a| \neq |b|$.

To compute A^{-1}, observe that A is secretly a block diagonal matrix composed of n copies of $\left[\begin{array}{cc} a & b \\ b & a \end{array}\right]$. The blocks are not in consecutive rows and columns, but in positions $\{i, 2n+1-i\} \times \{i, 2n+1-i\}$ for $i = 1, 2, \dots, n$. Since

$$\left[\begin{array}{cc} a & b \\ b & a \end{array}\right]^{-1} = \frac{1}{a^2 - b^2} \left[\begin{array}{cc} a & -b \\ -b & a \end{array}\right]$$

it follows that

$$\boxed{A^{-1} = \frac{1}{a^2-b^2} \left[\begin{array}{cccccccc} a & & & & & & & -b \\ & a & & & & & -b & \\ & & \ddots & & & \iddots & & \\ & & & a & -b & & & \\ & & & -b & a & & & \\ & & \iddots & & & \ddots & & \\ & -b & & & & & a & \\ -b & & & & & & & a \end{array}\right]}$$

when $|a| \neq |b|$.

Problem 4. (10 points) This problem has two parts.

(a) Determine the values of the constants $a, b \in \mathbb{R}$ such that the linear system

$$\begin{cases} x_1 + ax_2 + 2x_3 &= 2 \\ 4x_1 - 8x_2 + 8x_3 &= b \end{cases}$$

has (1) a unique solution, (2) infinitely many solutions, or (3) no solution.

Find the general solution in terms of a and b in cases (1) and (2).

Solution to part (a):

The augmented matrix of the given system is

$$A = \begin{bmatrix} 1 & a & 2 & 2 \\ 4 & -8 & 8 & b \end{bmatrix} \text{ which is row equivalent to } \begin{bmatrix} 1 & a & 2 & 2 \\ 0 & -8-4a & 0 & b-8 \end{bmatrix}.$$

If $-8-4a = 0 \neq b-8$ then there is a pivot in the last column so there are no solutions. Thus $\boxed{\text{there are no solutions if } a = -2 \text{ and } b \neq 8}$.

If $a = -2$ and $b = 8$ then

$$\mathsf{RREF}(A) = \begin{bmatrix} 1 & -2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so there are infinitely many solutions, each of the form

$$\boxed{(x_1, x_2, x_3) = (2 + 2y - 2z, y, z) \text{ where } y, z \in \mathbb{R} \text{ are arbitrary}}.$$

If $a \neq -2$ then further row reduction gives

$$\begin{bmatrix} 1 & a & 2 & 2 \\ 0 & -8-4a & 0 & b-8 \end{bmatrix} \to \begin{bmatrix} 1 & a & 2 & 2 \\ 0 & 1 & 0 & \frac{8-b}{8+4a} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & \frac{16+ab}{8+4a} \\ 0 & 1 & 0 & \frac{8-b}{8+4a} \end{bmatrix} = \mathsf{RREF}(A).$$

This means there are again infinitely many solutions, each of the form

$$\boxed{(x_1, x_2, x_3) = \left(\frac{16+ab}{8+4a} - 2z, \frac{8-b}{8+4a}, z\right) \text{ where } z \in \mathbb{R} \text{ is arbitrary}}.$$

(b) Suppose A is a 3×3 matrix with all real entries.

The complex number $\lambda = 3 - 2i$ is an eigenvalue of A and $\det(A) = 65$.

What is the trace of A?

Explain how you found your answer to receive full credit.

Solution to part (b):

A must have two other eigenvalues of the form $3 + 2i$ and $a + bi$ for some $a, b \in \mathbb{R}$. But then $\det(A) = (3-2i)(3+2i)(a+bi) = (9+4)(a+bi) = 13a + 13bi = 65$ so $a = 5$ and $b = 0$. Therefore $\mathrm{tr}(A) = (3-2i) + (3+2i) + 5 = \boxed{11}$.

Problem 5. (15 points) This problem has five parts.

(a) Give an example of a diagonal square matrix that is not invertible.

Any square zero matrix will do, such as the 1×1 matrix $\left[\begin{array}{c} 0 \end{array}\right]$.

(b) Give an example of a diagonalizable square matrix that is not diagonal.

An $n \times n$ triangular matrix with all distinct diagonal entries has n distinct eigenvalues so is diagonalizable. So one example is $\left[\begin{array}{cc} 1 & 1 \\ 0 & 2 \end{array}\right]$.

(c) Give an example of a triangular square matrix that is not diagonalizable.

We saw in class that any triangular-but-not-diagonal square matrix with equal diagonal entries is not diagonalizable. So any example is $\left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right]$.

(d) Give an example of an invertible square matrix that is not triangular.

Every permutation matrix is invertible, but only the identity matrix is triangular. So one example is $\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$.

(e) Give an example of an orthogonal 2×2 matrix that is not a rotation matrix.

The matrix $\left[\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right]$ is orthogonal: its columns are orthonormal. It is not a rotation matrix since every rotation matrix $\left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$ has equal diagonal entries.

Problem 6. (15 points) This question has three parts.

(a) Find the orthogonal projection of the vector $v = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$ onto the subspace

$$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + y + z = 0 \right\}.$$

Show all steps in your calculations to receive full credit.

Solution to part (a):

A basis for H is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ which we can convert to an orthogonal basis by setting

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and

$$x_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}.$$

Then the projection is

$$\begin{aligned} \mathrm{proj}_H(v) &= \frac{v \bullet x_1}{x_1 \bullet x_1} x_1 + \frac{v \bullet x_2}{x_2 \bullet x_2} x_2 \\ &= \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{-5/2}{3/2} \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} + \begin{bmatrix} 5/6 \\ -5/3 \\ 5/6 \end{bmatrix} = \begin{bmatrix} 8/6 \\ -5/3 \\ 2/6 \end{bmatrix} \end{aligned}$$

which simplifies to the answer

$$\boxed{\mathrm{proj}_H(v) = \begin{bmatrix} 4/3 \\ -5/3 \\ 1/3 \end{bmatrix}}.$$

(b) Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line of best fit for the data points $(x, y) = (-1, 0)$, $(0, 1)$, $(1, 2)$, $(2, 4)$.

Sketch a plot of the data points along with your line of best fit.

Solution to part (b):

We find β_0 and β_1 as the least-squares solution to the equation

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}.$$

The least-squares solution to this equation are the exact solutions to

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}.$$

Since $\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{24-4} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$ the unique solution is

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 10 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 11 \\ 13 \end{bmatrix}$$

and the line of best fit is

$$\boxed{y = 1.1 + 1.3x}.$$

(For full points, a picture of this line and the data points should also be included.)

(c) Find $x, y \in \mathbb{R}$ that minimize the distance between $\begin{bmatrix} 2x \\ 0 \\ 2x \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ y \\ 2y \\ y \end{bmatrix}$.

Solution to part (c):

Minimizing the distance means to minimize

$$\left\| \begin{bmatrix} 2x \\ 0 \\ 2x \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ y \\ 2y \\ y \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2x \\ 0 \\ 2x \\ y \end{bmatrix} - \begin{bmatrix} 2 \\ y \\ 2y \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2x \\ -y \\ 2x-2y \\ y \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\|$$
$$= \left\| \begin{bmatrix} 2 & 0 \\ 0 & -1 \\ 2 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\|.$$

But any vector that minimize this quantity is a least-squares solution

$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \\ 2 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Such least-squares solutions are the exact solutions to

$$\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \\ 2 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} 8 & -4 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

This equation has a unique solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \frac{1}{32} \begin{bmatrix} 6 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \frac{1}{32} \begin{bmatrix} 28 \\ 24 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

so the answer is $\boxed{x = 7/8}$ and $\boxed{y = 3/4}$.

Problem 7. (15 points)

Define $\mathbb{R}^{2\times 2}$ to be the set of all 2×2 matrices with all real entries.

The set $\mathbb{R}^{2\times 2}$ is a vector space. Define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by the formula

$$T(A) = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} A \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}.$$

This is a linear function.

(a) Find a basis for the subspace $\mathsf{range}(T) = \{T(A) : A \in \mathbb{R}^{2\times 2}\}$.

Solution to part (a):

The first thing to do is to compute a more explicit formula

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2a & 2b \\ 2a+2c & 2b+2d \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4(a+b) & 2(a+b) \\ 4(a+b+c+d) & 2(a+b+c+d) \end{bmatrix}.$$

We see from that $T(A)$ always has the form $\begin{bmatrix} 4x & 2x \\ 4y & 2y \end{bmatrix}$ where $x = a+b$ and $y = a+b+c+d$ can be any two real numbers. So a basis for $\mathsf{range}(T)$ is $\boxed{\begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 4 & 2 \end{bmatrix}}$.

(b) Find a basis for the subspace $\mathsf{kernel}(T) = \left\{A \in \mathbb{R}^{2\times 2} : T(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$.

Solution to part (b):

To be in $\mathsf{kernel}(T)$, the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ must have $a + b = 0$ and $a + b + c + d = c + d = 0$, meaning that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -b & b \\ -d & d \end{bmatrix} = b\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

So a basis for $\mathsf{kernel}(T)$ is $\boxed{\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}}$.

(c) Find all **nonzero** numbers $\lambda \in \mathbb{R}$ such that $T(A) = \lambda A$ for some nonzero matrix $A \in \mathbb{R}^{2\times 2}$. For each of these nonzero eigenvalues λ, compute a basis for the subspace $\{A \in \mathbb{R}^{2\times 2} : T(A) = \lambda A\}$.

Solution to part (c):

If the matrix

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 4(a+b) & 2(a+b) \\ 4(a+b+c+d) & 2(a+b+c+d) \end{bmatrix}$$

is a nonzero scalar multiple of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then we must have $a = 2b$ and $c = 2d$. For

$$T\left(\begin{bmatrix} 2b & b \\ 2d & d \end{bmatrix}\right) = \begin{bmatrix} 12b & 6b \\ 12(b+d) & 6(b+d) \end{bmatrix}$$

to equal

$$\lambda \begin{bmatrix} 2b & b \\ 2d & d \end{bmatrix} = \begin{bmatrix} 2\lambda b & \lambda b \\ 2\lambda d & \lambda d \end{bmatrix}$$

we must have $\lambda = 6$ and $b = 0$. The only possibility for λ is $\boxed{\lambda = 6}$ and then a basis for the corresponding eigenspace is $\boxed{\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}}$.

Problem 8. (15 points) This question has three parts.

(a) Compute the singular values of the matrix $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$.

Solution to part (a):

A shortcut to solving this part is to realize that A and $A^\top$ have the same nonzero singular values, that is, the nonzero singular values of A are the square roots of the eigenvalues of $AA^\top$. We have

$$AA^\top = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

so $\det(AA^\top - xI) = (17-x)^2 - 8^2 = (25-x)(9-x)$ has roots $\lambda_1 = 25$ and $\lambda_2 = 9$. So that first two singular values of A are $\boxed{\sigma_1 = 5}$ and $\boxed{\sigma_2 = 3}$ while the third is $\boxed{\sigma_3 = 0}$ since A has rank two.

(b) Suppose $A = U\Sigma V^\top$ is a singular value decomposition with

$$U = \begin{bmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 71 & 0 & 0 \\ 0 & 31 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad V^\top = \begin{bmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ v_7 & v_8 & v_9 \end{bmatrix}.$$

Find a basis for $\mathsf{Col}(A)$ and a basis for $\mathsf{Nul}(A)$.

Solution to part (b):

Since $V^\top$ is invertible, $\{V^\top x : x \in \mathbb{R}^3\} = \mathbb{R}^3$, so

$$\begin{aligned} \mathsf{Col}(A) &= \{U\Sigma V^\top x : x \in \mathbb{R}^3\} \\ &= \{U\Sigma x : x \in \mathbb{R}^3\} \\ &= \left\{ U \begin{bmatrix} 71x_1 \\ 72x_2 \\ 0 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\} = \left\{ U \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\} \\ &= \left\{ x_1 \begin{bmatrix} u_1 \\ u_4 \\ u_7 \end{bmatrix} + x_2 \begin{bmatrix} u_2 \\ u_5 \\ u_8 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\} = \mathbb{R}\text{-span} \left\{ \begin{bmatrix} u_1 \\ u_4 \\ u_7 \end{bmatrix}, \begin{bmatrix} u_2 \\ u_5 \\ u_8 \end{bmatrix} \right\}. \end{aligned}$$

Thus $\boxed{\begin{bmatrix} u_1 \\ u_4 \\ u_7 \end{bmatrix}, \begin{bmatrix} u_2 \\ u_5 \\ u_8 \end{bmatrix}}$ is a basis for $\mathsf{Col}(A)$. The null space of A has dimension $3 - \operatorname{rank}(A) = 1$. A nonzero vector in $\mathsf{Nul}(A)$ is $\boxed{\begin{bmatrix} v_7 \\ v_8 \\ v_9 \end{bmatrix}}$ since

$$A \begin{bmatrix} v_7 \\ v_8 \\ v_9 \end{bmatrix} = U\Sigma V^\top \begin{bmatrix} v_7 \\ v_8 \\ v_9 \end{bmatrix} = U\Sigma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = U \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

and this vector provides a basis.

(c) Let $\mathbb{D}^2$ be the set of vectors $v \in \mathbb{R}^2$ with $\|v\| = 1$.

Suppose A is a 2×2 matrix with

$$\min\{\|Av\| : v \in \mathbb{D}^2\} = 20 \quad \text{and} \quad \max\{\|Av\| : v \in \mathbb{D}^2\} = 22.$$

Assume that $A \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \end{bmatrix}$.

Draw a picture of the region $\{Av : v \in \mathbb{D}^2\}$ in $\mathbb{R}^2$.

Then determine all possible values for A.

Solution to part (c):

The matrix A transforms the unit vector $\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$ to the length 20 vector $\begin{bmatrix} 0 \\ 20 \end{bmatrix}$.

The region $\{Av : v \in \mathbb{D}^2\}$ is an ellipse whose radii have lengths 22 and 20. It follows that the shorter radii of this ellipse are the vectors $\begin{bmatrix} 0 \\ 20 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -20 \end{bmatrix}$ while the longer radii are $\begin{bmatrix} 22 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -22 \\ 0 \end{bmatrix}$.

The unit vector $\begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$ must be mapped by A to one of the latter two vectors. Therefore A must have one of the two singular value decompositions

$$A = \begin{bmatrix} \pm 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 22 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix}^\top$$

which simplifies to

$$\boxed{A = \begin{bmatrix} 88/5 & -66/5 \\ 60/5 & 80/5 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} -88/5 & 66/5 \\ 60/5 & 80/5 \end{bmatrix}}.$$