FINAL EXAMINATION SOLUTIONS – MATH 2121, FALL 2023

Problem 1. (20 points) Warmup: definitions and core concepts.

Provide short answers to the following questions.

(1) What is the definition of a **linear** function $f : \mathbb{R}^n \to \mathbb{R}^m$?

A function with $f(u+v) = f(u)+f(v)$ and $f(cv) = cf(v)$ for all $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

(2) How many solutions can a linear system have?

0, 1, or infinitely many.

(3) What is the definition of a **subspace** of a vector space?

A subset V of a vector space is a subspace if it is nonzero such that if $u, v \in V$ and $c \in \mathbb{R}$ then $u + v \in V$ and $cv \in V$.

(4) How can you compute the rank of an $m \times n$ matrix A?

Row reduce A and count the number of pivot columns.

(5) How can you compute the inverse of an invertible $n \times n$ matrix A?

Row reduce the $n \times 2n$ matrix $\left[\begin{array}{cc} A & I \end{array}\right]$ and then take the right $n \times n$ submatrix of the result.

(6) What region of $\mathbb{R}^2$ always has area equal to $\pm \det \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$? Draw and label a picture that represents this region.

One answer is the parallelogram with sides $\left[\begin{array}{c} a \\ c \end{array}\right]$ and $\left[\begin{array}{c} c \\ d \end{array}\right]$.

(7) The least-squares solutions to $Ax = b$ are the exact solutions to what matrix equation?

$A^\top Ax = A^\top b$.

(8) What $n \times n$ matrices have n orthonormal eigenvectors?

Symmetric matrices.

(9) What is the definition of the **singular values** of a matrix A?

The square roots of the eigenvalues of $A^\top A$.

(10) Suppose A is a $2 \times n$ matrix with a singular value decomposition

$$A = U\Sigma V^\top.$$

Assume $\operatorname{rank} A = 2$. Describe the shape

$$\{Av \in \mathbb{R}^2 : v \in \mathbb{R}^n \text{ with } \|v\| = 1\}$$

and explain how this shape is related to the matrices U and Σ.

The shape is an ellipse, centered at the origin, with radius vectors whose lengths are the diagonal entries of Σ, and whose directions are given by the columns of U.

Problem 2. (10 points) Suppose a and b are real numbers. Consider the lines

$$L_1 = \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 : v_2 = av_1 \right\} \quad \text{and} \quad L_2 = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2 : w_2 = bw_1 \right\}.$$

For which values of a and b is there exactly one way of writing

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} = v + w$$

where $v \in L_1$ and $w \in L_2$? Find a formula for v and w in this case.

Solution:

As long as $\boxed{a \neq b}$, so that the lines are not parallel, there will be a unique way of writing the given vector as $v + w$ where $v \in L_1$ and $w \in L_2$. To find these vectors, we want to find $v_1, w_1 \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix} \begin{bmatrix} v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ av_1 \end{bmatrix} + \begin{bmatrix} w_1 \\ bw_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}.$$

We can solve for these numbers by row reducing

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ a & b & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & b-a & 6-2a \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{6-2a}{b-a} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{2b-2a}{b-a} - \frac{6-2a}{b-a} \\ 0 & 1 & \frac{6-2a}{b-a} \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & \frac{2b-6}{b-a} \\ 0 & 1 & \frac{6-2a}{b-a} \end{array}\right].$$

The first entry in the last column is v_1 and the second entry is w_1 so

$$\boxed{v = \frac{2b-6}{b-a} \begin{bmatrix} 1 \\ a \end{bmatrix} \quad \text{and} \quad w = \frac{6-2a}{b-a} \begin{bmatrix} 1 \\ b \end{bmatrix}}$$

Problem 3. (10 points) Does there exist a pair of 2×2 matrices A and B with all real entries such that A has only one real eigenvalue, B has only one real eigenvalue, and $A + B$ has two distinct real eigenvalues?

Find an example or explain why none exists.

Solution:

The matrices $\boxed{A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$ and $\boxed{B = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}}$ both have only one real eigenvalue since they are triangular, but $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has two real eigenvalues 1 and -1 since its characteristic polynomial is $x^2 - 1 = (x-1)(x+1)$.

Problem 4. (10 points) Let V be the vector space of polynomials $f = ax^2 + bx + c$ of degree at most two with all coefficients $a, b, c \in \mathbb{R}$. Given $f, g \in V$ let

$$f \bullet g = \int_0^1 fg.$$

Here we define integration $\int_0^1$ to be the linear operation on polynomials with

$$\int_0^1 x^n = 1/(n+1).$$

Find a basis for V that is orthonormal using this definition of inner product.

In other words, if $d = \dim V$, then find a basis $f_1, f_2, \dots, f_d$ for V such that

$$f_i \bullet f_i = 1 \quad \text{and} \quad f_i \bullet f_j = 0 \quad \text{for all } i, j \in \{1, 2, \dots, d\} \text{ with } i \neq j.$$

Solution:

A basis for V is $1, x, x^2$ but these are not orthonormal. We use the Gram-Schmidt process adapted to this slightly unusual setting to get an orthogonal basis:

$f_1 = 1,$

$$f_2 = x - \frac{x \bullet f_1}{f_1 \bullet f_1} f_1 = x - \frac{\int_0^1 x}{\int_0^1 1} 1 = x - \tfrac{1}{2},$$

$$f_3 = x^2 - \frac{x^2 \bullet f_1}{f_1 \bullet f_1} f_1 - \frac{x^2 \bullet f_2}{f_2 \bullet f_2} f_2 = x^2 - \frac{\int_0^1 x^2}{\int_0^1 1} 1 - \frac{\int_0^1 (x^3 - x^2/2)}{\int_0^1 (x^2 - x + 1/4)} (x - \tfrac{1}{2})$$
$$= x^2 - \tfrac{1}{3} - \frac{1/4 - 1/6}{1/3 - 1/2 + 1/4} (x - \tfrac{1}{2}) = x^2 - x + \tfrac{1}{6}.$$

These vectors are not yet orthonormal. We want to compute

$$\tfrac{1}{\sqrt{f_1 \bullet f_1}} f_1, \quad \tfrac{1}{\sqrt{f_2 \bullet f_2}} f_2, \quad \tfrac{1}{\sqrt{f_3 \bullet f_3}} f_3.$$

Observe that $f_1 \bullet f_1 = 1$ and

$$f_2 \bullet f_2 = \int_0^1 (x^2 - x + 1/4) = 1/3 - 1/2 + 1/4 = 1/12$$

and

$$\begin{aligned} f_3 \bullet f_3 &= \int_0^1 (x^4 - 2x^3 + 4/3x^2 - x/3 + 1/36) \\ &= 1/5 - 1/2 + 4/9 - 1/6 + 1/36 \\ &= 1/5 - 18/36 + 16/36 - 6/36 + 1/36 \\ &= 1/5 - 7/36 = 36/180 - 35/180 = 1/180. \end{aligned}$$

So the final answer is

$$\boxed{1, \quad \sqrt{3}(2x-1), \quad \sqrt{5}(6x^2 - 6x + 1)}$$

Problem 5. (10 points) Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a one-to-one linear transformation with standard matrix A. If all we know is that $n \in \{1, 2, 3\}$ and $m \in \{1, 2, 3\}$, then what matrices could occur as $\mathsf{RREF}(A)$?

Describe the possibilities for $\mathsf{RREF}(A)$ in as much detail as you can.

Solution:

$\mathsf{RREF}(A)$ must be an $m \times n$ matrix with a pivot in every column, and $n \leq m$. So we could have

$$\boxed{\left[\begin{array}{c} 1 \end{array}\right], \quad , \left[\begin{array}{c} 1 \\ 0 \end{array}\right], \quad \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right], \quad \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right], \quad \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}\right], \quad \text{or} \quad \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]}$$

Problem 6. (10 points) Suppose $A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 0 \\ 2 & 3 & 4 \end{bmatrix}$ and $v \in \mathbb{R}^3$.

Define w_i to be the determinant of A with column i replaced by v.

Does any matrix B exist with $Bv = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ for all choices of $v \in \mathbb{R}^3$?

Compute the matrix B or explain why no such B exists.

Solution:

We want to find a matrix B with

$$Bv = \begin{bmatrix} \det \begin{bmatrix} v_1 & 0 & 2 \\ v_2 & 2 & 0 \\ v_3 & 3 & 4 \end{bmatrix} \\ \\ \det \begin{bmatrix} 2 & v_1 & 2 \\ 1 & v_2 & 0 \\ 2 & v_3 & 4 \end{bmatrix} \\ \\ \det \begin{bmatrix} 2 & 0 & v_1 \\ 1 & 2 & v_2 \\ 2 & 3 & v_3 \end{bmatrix} \end{bmatrix}.$$

Such a matrix exists since the right hand is a linear function of v by the defining properties of the determinant. To compute B, we plug in $v = e_1, e_2, e_3$ to get the columns of the matrix. This gives

$$Be_1 = \begin{bmatrix} 8 \\ -4 \\ -1 \end{bmatrix} \quad Be_2 = \begin{bmatrix} 6 \\ 4 \\ -6 \end{bmatrix} \quad Be_3 = \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}$$

so $\boxed{B = \begin{bmatrix} 8 & 6 & -4 \\ -4 & 4 & 2 \\ -1 & -6 & 4 \end{bmatrix}}$.

Problem 7. (10 points) Suppose $u, v, w \in \mathbb{R}^3$ and $A = \begin{bmatrix} u & v & w \end{bmatrix}$.

If $\det(A) = 30$ then what is

$$\det \begin{bmatrix} u + 2v - 3w & v + w & 2u + v + 2w \end{bmatrix}?$$

Justify your answer to receive full credit.

Solution:

We notice that

$$\begin{bmatrix} u + 2v - 3w & v + w & 2u + v + 2w \end{bmatrix} = \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ -3 & 1 & 2 \end{bmatrix}.$$

The determinant of the first matrix is $\det(A) = 30$ and the determinant of the second matrix is $1 - 0 + 10 = 11$ so the answer is their product which is $\boxed{330}$.

Problem 8. (10 points) Is the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

diagonalizable over the complex numbers?

If it is not, then explain why not. If it is, then find an invertible matrix P and a diagonal matrix D, possibly with complex entries, such that $A = PDP^{-1}$.

Solution:

The characteristic polynomial is $\det(A - xI) = -x(x^2) - 0 + 1 = 1 - x^3$. One root is $x = 1$ and the polynomial factors as $(1-x)(1+x+x^2)$. By the quadratic formula the other two roots are $\frac{-1\pm\sqrt{-3}}{2}$ or

$$\omega = \frac{-1+i\sqrt{3}}{2} \quad \text{and} \quad \lambda = \frac{-1-i\sqrt{3}}{2}.$$

These three roots are the eigenvalues of A. As they are three distinct complex numbers, A diagonalizable. Notice that

$$\omega^2 = \frac{1-2i\sqrt{3}-3}{4} = \frac{-2-2i\sqrt{3}}{4} = \lambda \quad \text{and} \quad \omega^3 = \omega\lambda = 1.$$

So the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \omega^2 \\ 1 \\ \omega \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$$

are eigenvectors with eigenvalues $1, \omega$, and ω^2 since

$$A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ a \\ b \end{bmatrix}.$$

So one final answer is

$$\boxed{P = \begin{bmatrix} 1 & \omega^2 & 1 \\ 1 & 1 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}}$$

Problem 9. (10 points) Let A be a symmetric $n \times n$ matrix with exactly one nonzero position in each row and exactly one nonzero position in each column.

Suppose the nonzero positions of A that are on or above the diagonal are

$$(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k).$$

In terms of this data, describe an orthogonal basis for $\mathbb{R}^n$ that consists of eigenvectors for A. Be as concrete as possible.

Solution:

For each $t = 1, 2, \ldots, k$, if $i_t = j_t$ then add the standard basis vector e_{i_t} and if $i_t < j_t$ then add the pair of vectors $e_{i_t} + e_{j_t}$ and $e_{i_t} - e_{j_t}$. This will result in n eigenvectors that are orthogonal eigenvectors for A, hence an orthogonal basis for $\mathbb{R}^n$ that consists of eigenvectors for A.

Problem 10. (10 points) Suppose A is a 3×3 matrix with all real entries, whose eigenvalues include the complex numbers 2 and $1-i$. Find a polynomial formula for the function $f(x) = \det(A^{-1} - xI)$ and compute $f(5)$.

Solution:

The complex conjugate $1+i$ must also be an eigenvalue of A. So A has three distinct eigenvalues so is diagonalizable, and can be written as

$$A = P \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1-i & 0 \\ 0 & 0 & 1+i \end{bmatrix} P^{-1}.$$

Therefore

$$A^{-1} = P \begin{bmatrix} 2^{-1} & 0 & 0 \\ 0 & (1-i)^{-1} & 0 \\ 0 & 0 & (1+i)^{-1} \end{bmatrix} P^{-1} = P \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & (1+i)/2 & 0 \\ 0 & 0 & (1-i)/2 \end{bmatrix} P^{-1}$$

and so

$$A^{-1} - xI = P \begin{bmatrix} 1/2 - x & 0 & 0 \\ 0 & (1+i)/2 - x & 0 \\ 0 & 0 & (1-i)/2 - x \end{bmatrix} P^{-1}.$$

Hence

$$f(x) = \det(A^{-1} - xI) = (1/2 - x)(\tfrac{1+i}{2} - x)(\tfrac{1-i}{2} - x) = \boxed{(1/2 - x)(1/2 - x + x^2)}$$

and $f(5) = (1/2 - 5)(1/2 - 5 + 25) = \frac{(1-10)(1-10+50)}{4} = \frac{(-9)(41)}{4} = \boxed{\dfrac{-369}{4}}$.

Problem 11. (10 points) Suppose $u, v, w \in \mathbb{R}^n$ are linearly independent vectors.

For which values of $c \in \mathbb{R}$ are the three vectors

$$5w - 3u, \quad 5u + 3v + 4w, \quad 6v - 2u + cw$$

linearly dependent?

Solution:

The three vectors are linearly dependent if and only if the columns of

$$\begin{bmatrix} -3 & 5 & -2 \\ 0 & 3 & 6 \\ 5 & 4 & c \end{bmatrix}$$

are linearly dependent, i.e., the matrix is not invertible. The determinant is

$-3(3c - 24) - 5(-30) - 2(-15) = -9c + 72 + 150 + 30 = -9c + 252 = -9(c - 28)$

which is zero if $\boxed{c = 28}$.

Problem 12. (10 points) Suppose $u, v, w \in \mathbb{R}^5$.

What are the possible values of $\mathrm{rank}(uu^\top + vv^\top + ww^\top)$?

Justify your answer to receive full credit.

Solution:

The rank is either $\boxed{0, 1, 2, \text{or } 3}$. The rank is ≤ 3 because the column space of $A = uu^\top + vv^\top + ww^\top$ is contained in $\mathbb{R}$-span$\{u, v, w\}$. We can achieve any of the possible ranks by taking $u = v = w = 0$ (rank 0), or $u = e_1$ and $v = w = 0$ (rank 1), or $u = e_1$ and $v = e_2$ and $w = 0$ (rank 2), or $u = e_1$ and $v = e_2$ and $w = e_3$ (rank 3).

Problem 13. (10 points) A is a 2×2 matrix and $-1 < \lambda < 1$ is a real number with

$$A \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

Compute A and $\lim_{n\to\infty} A^n$.

The entries in your answer for A should be expressions involving λ.

Solution:

From the given information we know that $A = PDP^{-1}$ for $P = \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$. Then $P^{-1} = \frac{1}{-8} \begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix} = \frac{1}{8} P$ so

$$\boxed{A = \frac{1}{8} \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}}$$

and

$$\boxed{\lim_{n\to\infty} A^n = \frac{1}{8} \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}.$$

Problem 14. (10 points) Suppose

$$v = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find a matrix A such that if $x \in \mathbb{R}^4$ then

$$Ax = (\text{ the vector in } \mathbb{R}\text{-span}\{v,w\} \text{ that is as close to } x \text{ as possible }).$$

Solution:

An orthogonal basis for $V = \mathbb{R}\text{-span}\{v,w\}$ is

$$v = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad u = 3(w - \frac{w \bullet v}{v \bullet v} v) = 3\left(\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 4 \\ 2 \\ 3 \end{bmatrix}.$$

The vector in V that is as close to x as possible is

$$\text{proj}_V(x) = \frac{x \bullet v}{v \bullet v} v + \frac{x \bullet u}{u \bullet u} u = \begin{bmatrix} \frac{v}{v \bullet v} & \frac{u}{u \bullet u} \end{bmatrix} \begin{bmatrix} v \bullet x \\ u \bullet x \end{bmatrix} = \begin{bmatrix} v & u \end{bmatrix} \begin{bmatrix} \frac{1}{v \bullet v} & 0 \\ 0 & \frac{1}{u \bullet u} \end{bmatrix} \begin{bmatrix} v^\top \\ u^\top \end{bmatrix} x.$$

As $v \bullet v = 3$ and $u \bullet u = 4 + 16 + 4 + 9 = 33$ the desired matrix is

$$\boxed{A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \\ -1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1/33 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & 4 & 2 & 3 \end{bmatrix}}$$

Problem 15. (10 points) A is an invertible $n \times n$ matrix with at least one real eigenvalue. There is no nonzero vector $v \in \mathbb{R}^n$ such that $Av = v$. If 3 is the only eigenvalue of $A + 2A^{-1}$ then what number must be an eigenvalue of A? Justify your answer to receive full credit.

Solution:

Suppose $Av = \lambda v$ where v is nonzero and $\lambda \in \mathbb{R}$. Such a vector exists by hypothesis and we know that $\lambda \neq 1$. Then $A^{-1}v = \lambda^{-1}v$ so

$$(A + 2A^{-1})v = Av + 2A^{-1}v = (\lambda + 2\lambda^{-1})v.$$

As 3 is the only eigenvalue, we must have $\lambda + 2\lambda^{-1} = 3$ which is equivalent to

$$\lambda^2 + 2 = 3\lambda.$$

The quadratic formula gives two solutions to this equation: $\lambda = 1$ and $\lambda = 2$. As $\lambda \neq 1$ the number $\boxed{2}$ must be an eigenvalue of A.

Problem 16. (10 points) Does there exist an invertible 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

for which the determinant of every 2×2 submatrix involving **consecutive** rows and columns is zero? In other words, with

$$\det \begin{bmatrix} a_{ij} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{bmatrix} = 0$$

for all $i \in \{1, 2\}$ and $j \in \{1, 2\}$?

Find an example or explain why none exists.

Solution:

If all consecutive 2×2 submatrices have determinant zero then

$$\det A = -a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}.$$

For this determinant to be nonzero we must have $a_{12} \neq 0$ and the vectors $\begin{bmatrix} a_{21} \\ a_{31} \end{bmatrix}$ and $\begin{bmatrix} a_{23} \\ a_{33} \end{bmatrix}$ must not be scalar multiples of each other. But both vectors are scalar multiples of $\begin{bmatrix} a_{22} \\ a_{32} \end{bmatrix}$, so this is only possible if $\begin{bmatrix} a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

But then $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} a_{12} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ are scalar multiples of each other, which is only possible if $a_{21} = 0$.

Likewise $\begin{bmatrix} a_{12} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$ are scalar multiples of each other, which is only possible if $a_{23} = 0$.

But if $a_{21} = a_{23} = 0$ then $\begin{bmatrix} a_{21} \\ a_{31} \end{bmatrix}$ and $\begin{bmatrix} a_{23} \\ a_{33} \end{bmatrix}$ are indeed scalar multiples of each other. Thus it is not possible to have $\det A \neq 0$ so $\boxed{\text{no such matrix exists}}$.

Problem 17. (10 points) What is the largest possible number that can occur as the determinant of a 3×3 matrix with all entries in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$? What matrix achieves this determinant?

Solution:

The maximum possible determinant is $2 \cdot 9^3$. One matrix that achieves this is

$$\left[\begin{array}{ccc} 0 & 9 & 9 \\ 9 & 0 & 9 \\ 9 & 9 & 0 \end{array}\right].$$

There are others.