This document is a **transcript** of the lecture, with extra summary and vocabulary sections for your convenience. Due to time constraints, the notes may sometimes only contain limited illustrations, proofs, and examples; for a more thorough discussion of the course content, **consult the textbook**.

Summary

Quick summary of today's notes. Lecture starts on next page.

- A *linear equation* in variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form
$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$
where the a_i's and b are numbers. Like
$$4x_1 + 5x_2 - \pi x_3 = -1 \quad \text{or} \quad x_1 = x_2 + 3 \quad \text{or} \quad 0 = 0 \quad \text{or even} \quad 0 = 1,$$
but NOT $x_1^2 + x_2^2 = 1$ or $|x_1| + 3x_2 = 0$ or $\sin(x_1) = \sqrt{2}/2$ or $2^{x_1+x_2} = 4$.
- A *linear system* is a list of linear equations.
- A *solution* to a linear system is an assignment of values to variables that make all equations in the system simultaneously true.

  The linear system with two equations $x_1 + x_2 = 7$ and $x_2 - x_1 = 1$ has a solution given by $(x_1, x_2) = (3, 4)$. Two linear systems are *equivalent* if they have the same solutions.
- Any linear system has 0, 1, or infinitely many solutions. If a linear system has two different solutions then it has infinitely many. If a linear system has no solutions then it is called *inconsistent*.
- A *matrix* is a rectangular array of numbers like $\left[\begin{array}{c}\sqrt{2}\end{array}\right]$ or $\left[\begin{array}{cc}1.1 & -1.1\\ 1.1 & 1.2\end{array}\right]$ or $\left[\begin{array}{ccc}1 & 7 & -1\\ 0 & 4 & 3\end{array}\right]$.
- A matrix A with m rows and n columns is said to be "$m \times n$" or "m-by-n."

  The rows of A are indexed top to bottom by the numbers $1, 2, 3, \ldots, m$ while the columns are indexed left to right by the numbers $1, 2, 3, \ldots, n$. The entry in row i and column j often denoted A_{ij}, or called the entry of A in position (i, j).
- There are two important matrices associated to a linear system: the *coefficient matrix* and the *augmented matrix*. These are best defined by example:
$$\underbrace{\begin{aligned} 3x_1 + x_3 &= 8 \\ x_2 - x_3 &= 0 \\ 5x_1 + 4x_2 + 2x_3 &= 1 \end{aligned}}_{\text{linear system}} \quad \rightsquigarrow \quad \underbrace{\left[\begin{array}{ccc} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 5 & 4 & 2 \end{array}\right]}_{\text{coefficient matrix}} \quad \text{and} \quad \underbrace{\left[\begin{array}{cccc} 3 & 0 & 1 & 8 \\ 0 & 1 & -1 & 0 \\ 5 & 4 & 2 & 1 \end{array}\right]}_{\text{augmented matrix}}.$$
- There are three *(elementary) row operations* we can perform on a matrix:

  (1) add a multiple of one row to another row,

  (2) multiply a row by a *nonzero* number,

  (3) swap two rows.

  For example:
$$\left[\begin{array}{cc} 0 & 0 \\ 1 & 2 \end{array}\right] \xrightarrow{\text{row op. (1)}} \left[\begin{array}{cc} 10 & 20 \\ 1 & 2 \end{array}\right] \xrightarrow{\text{row op. (2)}} \left[\begin{array}{cc} -5 & -10 \\ 1 & 2 \end{array}\right] \xrightarrow{\text{row op. (3)}} \left[\begin{array}{cc} 1 & 2 \\ -5 & -10 \end{array}\right].$$

  Two matrices are *row equivalent* if one can be transformed to the other by applying a finite sequence of these row operations.
- Linear systems with row equivalent augmented matrices have the same solutions (are *equivalent*).

1 Introduction

Check the course website

https://www.math.ust.hk/~emarberg/teaching/2023/Math2121/

for the syllabus and other course details.

Each lecture corresponds to one or more sections in the textbook.

Today's lecture corresponds to Section 1.1.

For a more detailed discussion of the topics in any particular lecture, see the textbook.

Throughout, we'll be using the following notation:

- $\mathbb{R}$ denotes the real numbers.
- $\mathbb{Q}$ denotes the rational numbers p/q.
- $\mathbb{Z}$ denotes the integers $\{\dots,-2,-1,0,1,2,\dots\}$.
- $\mathbb{N}$ denotes the nonnegative integers $\{0,1,2,\dots\}$.

Ellipsis ("...") notation: we write $a_1, a_2, \dots, a_7$ instead of the full list $a_1, a_2, a_3, a_4, a_5, a_6, a_7$.

We use the same convention to write $a_1, a_2, \dots, a_n$ even when n is a variable integer.

2 Systems of linear equations

Choose a positive integer n. Let $x_1, x_2, \dots, x_n$ be variables. Let $a_1, a_2, \dots, a_n, b$ be numbers in $\mathbb{R}$.

Unlike in calculus, where our favorite variables are x, y, z, in linear algebra we prefer $x_1, x_2, x_3, \dots$.

Definition. We refer to

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

as a *linear equation* in the variables $x_1, x_2, \dots, x_n$.

Notation. Another way of writing this equation is $\sum_{i=1}^n a_ix_i = b$.

The symbol "$\sum$" is the Greek letter sigma, for "sum."

There are many other ways of writing the same equation. For example:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n - b = 0,$$

$$b = a_1x_1 + a_2x_2 + \cdots + a_nx_n,$$

$$a_1x_1 + a_2x_2 + a_3x_3 = b - a_4x_4 - a_5x_5 - \cdots - a_nx_n,$$

and so forth. We consider all of these equations to be the same thing.

Example. The following are all linear equations in the variables x_1, x_2, x_3:

$$3x_1 = 2x_2, \qquad 3x_1 + \tfrac{4}{3}x_2 - \sqrt{2}x_3 = 7, \qquad 0 = 0, \qquad 0 = 1.$$

Even though the last two equations involve no variables, they have the form required of a linear equation. (The last equation is false, but a false equation is still an equation.)

The following are *not* linear equations in the variables x_1, x_2, x_3:

$$3x_1^2 + 4x_2 = 7, \qquad x_1x_2 = x_3, \qquad 2^{x_1} = x_2, \qquad \sqrt{x_1^2} = x_2.$$

The last equation almost looks linear, but remember that $\sqrt{x_1^2} = |x_1|$.

A *system of linear equations* or *linear system* is a list of linear equations.

Example.

$$\begin{aligned} 2x_1 - x_2 + \sqrt{3}x_3 &= 8 \\ x_1 - 4x_3 &= 8 \\ x_2 &= 0 \end{aligned}$$

is a linear system with three equations in the variables x_1, x_2, x_3.

When we discuss a linear system, there is always a set of associated variables, usually $x_1, x_2, \dots, x_n$ for some n. As we see in the preceding example, not every equation needs to involve every variable x_i. In fact, it could happen the some variable x_i appears in none of the equations.

Definition. A *solution* of a linear system in variables $x_1, x_2, \dots, x_n$ is a list of n numbers $(s_1, s_2, \dots, s_n)$ with the property that if we set $x_1 = s_1$, $x_2 = s_2$, ..., $x_n = s_n$ in our equations, we get all true statements.

A solution $(s_1, s_2, \dots, s_n)$ is *nonzero* if at least one number $s_i \neq 0$.

Two linear systems are *equivalent* if they have the same set of solutions.

Example. How many solutions can a linear system have?

1. The system
$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$
has one solution $(s_1, s_2) = (3, 2)$.
2. But the system
$$\begin{cases} x_1 - 2x_2 = -1 \\ 3x_1 - 6x_2 = -3 \end{cases}$$
has many solutions: $(s_1, s_2) = (1, 1)$ or $(3, 2)$ or $(5, 3)$ or ....
3. Whereas the system
$$\begin{cases} x_1 - 2x_2 = -1 \\ x_1 - 2x_2 = 0 \end{cases}$$
has no solutions.

Theorem. A linear system in two variables x_1 and x_2 has either 0, 1, or infinitely many solutions.

Proof by geometry. A solution to one equation $ax_1 + bx_2 = c$ represents a point on a line after we identify the pair of numbers (x_1, x_2) with a point in the Cartesian plane.

A solution to a system of 2-variable linear equations represents a point where the lines defined by the equations all intersect.

But a collection of lines all intersect at either 0 points (they don't have a common intersection), 1 point (the unique point of intersection) or at infinitely many points (in the case when the lines are all *the same line*, though they might come from different equations). $\square$

Proof by algebra. We just need to check that if our linear system has two different solutions, then it has infinitely many solutions.

Given a linear system, define the associated *homogeneous system* to be the linear system in which each equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is replaced by $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$.

For example the homogeneous system of

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases} \quad \text{is} \quad \begin{cases} x_1 - 2x_2 = 0 \\ -x_1 + 3x_2 = 0. \end{cases}$$

If (s_1, s_2) is a solution to our starting system and (t_1, t_2) is a solution to the associated homogeneous system, then $(s_1 + t_1, s_2 + t_2)$ is also a solution to our starting system.

This is because if $a_1s_1 + a_2s_2 = b$ and $a_1t_1 + a_2t_2 = 0$ then

$$a_1(s_1 + t_1) + a_2(s_2 + t_2) = (a_1s_1 + a_2s_2) + (a_1t_1 + a_2t_2) = b + 0 = b.$$

On the other hand, if the homogeneous system has a nonzero solution (t_1, t_2), then it has infinitely many solutions: the pairs $(2t_1, 2t_2)$, $(3t_1, 3t_2)$, $(4t_1, 4t_2)$, and so on are all solutions, since for example

$$\text{if } a_1t_1 + a_2t_2 = 0 \text{ then } a_1(4t_1) + a_2(4t_2) = 4(a_1t_1 + a_2t_2) = 4 \cdot 0 = 0.$$

Combining these observations means that if our starting system has a solution and the homogeneous system has a nonzero solution, then the starting system has infinitely many solutions.

Finally, observe that if our starting system has two different solutions (s_1, s_2) and (r_1, r_2), then

$$(t_1, t_2) = (s_1 - r_1, s_2 - r_2)$$

is a nonzero solution to the homogeneous system, since if $a_1s_1 + a_2s_2 = b$ and $a_1r_1 + a_2r_2 = b$ then

$$a_1(s_1 - r_1) + a_2(s_2 - r_2) = (a_1s_1 + a_2s_2) - (a_1r_1 + a_2r_2) = b - b = 0.$$

□

A linear system is *consistent* if it has at least one solution, and *inconsistent* if it has zero solutions.

Both the algebraic and geometric proofs generalize to any number of variables:

Theorem. A linear system (in any number of variables) has either 0, 1, or infinitely many solutions.

3 Matrices

A *matrix* is just a rectangular array of numbers, like these ones:

$$\begin{bmatrix} 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1/7 \\ 0.2 \\ 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 5 & 3 \\ 2 & \pi \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 7 & 6 & -4 & 3 \\ 2 & 1 & 1 & 0 \end{bmatrix}.$$

We denote a general matrix by

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix}$$

Here "A_{23}" is pronounced "A, two, three". This matrix is 3-by-4: it has 3 rows and 4 columns.

One says that a matrix A is m-by-n or $m \times n$ if it has m rows and n columns.

We usually write A_{ij} (pronounced "A, i, j") for the entry in the ith row and jth column of a matrix A.

Matrices are useful as a compact way of writing a linear system.

Consider the linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

Define the *coefficient matrix* of this system to be $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$.

In other words, the matrix A where A_{ij} is the coefficient of x_j in the ith equation.

The *augmented matrix* of the system is

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \quad \text{which we sometimes write as} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right].$$

We consider both of these to be the same matrix. The | on the right is just there to remind us that this is an augmented matrix rather than a coefficient matrix.

Exercise: how would you generalize this definition to any linear system?

4 Solving linear systems

We solve linear systems by adding equations together to cancel variables.

Next class, we will discuss a general algorithm for doing these kinds of cancellations that turns the augmented matrix of any linear system into an exact description of its solutions.

The following example contains some of the main ideas in this algorithm.

Example. To solve

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

we first add -5 times equation 1 to equation 3 to get

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{array}\right].$$

We then multiply equation 2 by 1/2 to get

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{array}\right].$$

We then add -10 times equation 2 to equation 3:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 30x_3 &= -30 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{array}\right].$$

Multiple equation 3 by 1/30:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

The augmented matrix of the last system if *triangular*: all entries in positions (i, j) with $i > j$ are zero.

Remember that i is the row, j is the column.

We can easily solve for x_1, x_2, x_3 from a triangular system, working from the bottom up:

- The last equation $\boxed{x_3 = -1}$ is already as simple as possible.
- Substitute into second equation: $x_2 - 4x_3 = x_2 - 4(-1) = 4 \Rightarrow \boxed{x_2 = 0}$.
- Substitute into first equation: $x_1 - 2x_2 + x_3 = x_1 - 2(0) + (-1) = 0 \Rightarrow \boxed{x_1 = 1}$.

Definition. In solving this system of equations, we performed the following *(elementary) row operations* on the augmented matrix of the system:

1. Replacement: replace one row by the sum of itself and a multiple of another row:
$$\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 501 & 602 \\ 5 & 6 \end{bmatrix}.$$
2. Scaling: multiply all entries in a row by a *nonzero* number: $\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ -500 & -600 \end{bmatrix}$.
3. Interchange: swap two rows: $\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}$.

Two matrices are *row equivalent* if one can be transformed to the other by a sequence of row operations.

Each row operation is reversible. (Exercise: why?)

Theorem. If the augmented matrices of two linear systems are row equivalent, then the systems are equivalent (i.e., have same solutions).

Proof. If we have two linear systems that differ by a single row operation, then a given list of numbers $(s_1, s_2, \ldots, s_n)$ is a solution to one system if and only if it is a solution to the other system.

Thus performing a sequence of row operations does not change the set of solutions to a linear system. □

5 Vocabulary

Keywords from today's lecture:

1. **Linear equation**.

   An equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ where n is a positive integer, $a_1, a_2, \ldots, a_n, b$ are numbers, and $x_1, x_2, \ldots, x_n$ are variables.

   Example: $3x_1 - \frac{1}{7}x_3 = x_4 + 5$.

2. **Linear system** or **system of linear equations**.

   A list of one or more linear equations.

   Example: $\begin{cases} x_1 + x_2 = 3 \\ 3x_2 - x_1 = 2 \end{cases}$

3. **Solution** to a linear system.

   A solution to one linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is a list of numbers $(s_1, s_2, \ldots, s_n)$ such that $a_1s_1 + a_2s_2 + \cdots + a_ns_n$ is equal to b. A solution to a linear system is a list of numbers that is simultaneously a solution to every equation in the system.

   Example: a solution to $\begin{cases} x_1 + x_2 = 3 \\ 3x_2 - x_1 = 2 \end{cases}$ is $(s_1, s_2) = (\frac{7}{4}, \frac{5}{4})$.

4. **Equivalent** linear systems.

   Two linear systems with the same sets of variables and same sets of solutions.

   Example: $\begin{cases} x_1 + x_2 = 3 \\ 3x_2 - x_1 = 2 \end{cases}$ and $\begin{cases} 2x_1 + 2x_2 = 6 \\ x_1 - 3x_2 + 2 = 0 \end{cases}$ are equivalent.

5. **Consistent** linear system.

   A linear system with at least one solution.

   Example: $\begin{cases} x_1 + x_2 = 3 \\ 3x_2 - x_1 = 2 \end{cases}$ is consistent.

6. **Inconsistent** linear system.

   A linear system with no solutions.

   Example: $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + 2x_2 = 4 \end{cases}$ is inconsistent.

7. **Matrix**.

   A rectangular array of numbers. A matrix A is $m \times n$ if it has m rows and n columns.

   We write A_{ij} for the entry of A is row i and column j.

   Example: $A = \begin{bmatrix} 0 & -1 & 2 \\ \sqrt{2} & 5 & 6 \end{bmatrix}$. This matrix is 2×3 and $A_{21} = \sqrt{2}$ while $A_{12} = -1$.

8. **Coefficient matrix** of a linear system.

   For a linear system m equations with n variables, the $m \times n$ matrix that records the coefficients of the variables.

   Example: $\begin{bmatrix} 0 & -1 & 2 \\ \sqrt{2} & 5 & 6 \end{bmatrix}$ is the coefficient matrix of $\begin{cases} -x_2 + 2x_3 = 3 \\ \sqrt{2}x_1 + 5x_2 + 6x_3 = 7. \end{cases}$

9. **Augmented matrix** of a linear system.

   For a linear system m equations with n variables, the $m \times (n+1)$ matrix that records the coefficients of the variables and the constant on the other side of each equation.

   Example: $\begin{bmatrix} 0 & -1 & 2 & 3 \\ \sqrt{2} & 5 & 6 & 7 \end{bmatrix}$ is the augmented matrix of $\begin{cases} -x_2 + 2x_3 = 3 \\ \sqrt{2}x_1 + 5x_2 + 6x_3 = 7. \end{cases}$

10. **Elementary row operator** on a matrix.

    One of the following operations on a matrix: replace one row by the sum of the row and a multiple of another row, multiply all entries in row by a fixed number, or swap two rows.

    Example: $\begin{bmatrix} 0 & -1 & 2 \\ \sqrt{2} & 5 & 6 \end{bmatrix} \to \begin{bmatrix} 2\sqrt{2} & 9 & 14 \\ \sqrt{2} & 5 & 6 \end{bmatrix}$

    Example: $\begin{bmatrix} 0 & -1 & 2 \\ \sqrt{2} & 5 & 6 \end{bmatrix} \to \begin{bmatrix} 0 & -1 & 2 \\ 5\sqrt{2} & 25 & 30 \end{bmatrix}$

    Example: $\begin{bmatrix} 0 & -1 & 2 \\ \sqrt{2} & 5 & 6 \end{bmatrix} \to \begin{bmatrix} \sqrt{2} & 5 & 6 \\ 0 & -1 & 2 \end{bmatrix}.$

11. **Row equivalent matrices**.

    Matrices that can be transformed to each other by a sequence of row operations.

    Example: $\begin{bmatrix} 0 & -1 & 2 \\ \sqrt{2} & 5 & 6 \end{bmatrix} \to \begin{bmatrix} 2\sqrt{2} & 9 & 14 \\ \sqrt{2} & 5 & 6 \end{bmatrix} \to \begin{bmatrix} 2\sqrt{2} & 9 & 14 \\ 5\sqrt{2} & 25 & 30 \end{bmatrix} \to \begin{bmatrix} 5\sqrt{2} & 25 & 30 \\ 2\sqrt{2} & 9 & 14 \end{bmatrix}$