This document is a **transcript** of the lecture, with extra summary and vocabulary sections for your convenience. Due to time constraints, the notes may sometimes only contain limited illustrations, proofs, and examples; for a more thorough discussion of the course content, **consult the textbook**.

Summary

Quick summary of today's notes. Lecture starts on next page.

- A *line of best fit* through data points $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)$ is an equation $y = \beta_0 + \beta_1 x$ where $\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \in \mathbb{R}^2$ is a least-squares solution to $Ax = b$ where $A = \begin{bmatrix} 1 & a_1 \\ 1 & a_2 \\ \vdots & \vdots \\ 1 & a_n \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.
- A matrix A is *symmetric* if $A^\top = A$. This can only hold if A is square. For example:

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & 8 \\ 0 & 8 & -7 \end{bmatrix}.$$

  If A is symmetric then so is A^2, A^3, A^4, etc.

  If A is symmetric and invertible then so is A^{-1}, A^{-2}, A^{-3}, etc.

  If A is symmetric and u and v are eigenvectors for A with **different** eigenvalues, then $u \bullet v = 0$.

- A list of vectors $u_1, u_2, \dots, u_p$ is *orthonormal* if $u_i \bullet u_i = 1$ and $u_i \bullet u_j = 0$ for all $i \neq j$.

  A square matrix P is invertible with $P^{-1} = P^\top$ if and only if its columns are orthonormal.

  An $n \times n$ matrix A is *orthogonally diagonalizable* if there is a diagonal matrix D and an invertible matrix P with $P^{-1} = P^\top$ such that $A = PDP^{-1}$.

- When $A = PDP^{-1}$ where D is diagonal and $P^{-1} = P^\top$, the diagonal entries of D are the eigenvalues of A, and the columns of P are an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors for A.

  Conversely, an $n \times n$ matrix A is orthogonally diagonalizable if and only if there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors for A.

- Surprising fact: all (complex) eigenvalues of a symmetric matrix $A = A^\top$ belong to $\mathbb{R}$.

  Surprising fact: an $n \times n$ matrix A is orthogonally diagonalizable if and only if $A = A^\top$.

  Much of this lecture is spent proving these facts.

- To orthogonally diagonalize a given $n \times n$ symmetric matrix A, you need to find an orthogonal basis of $\mathbb{R}^n$ consisting of eigenvectors $v_1, v_2, \dots, v_n$ for A.

  Once you find this, let $u_i = \frac{1}{\|v_i\|} v_i$ and $U = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}$.

  Then $A = UDU^\top$ where D is the diagonal matrix whose ith diagonal entry is the eigenvalue of v_i.

- To find the orthogonal basis of eigenvectors $v_1, v_2, \dots, v_n$ for A:
  1. Factor the characteristic polynomial of A to compute its eigenvalues.
  2. For each eigenvalue λ, do the usual row reduce procedure to find a basis for $\mathrm{Nul}(A - \lambda I)$.
  3. Apply the Gram-Schmidt process to convert your basis of $\mathrm{Nul}(A - \lambda I)$ to an orthogonal basis.
  4. Finally combine these orthogonal bases — the combined list of vectors will still be orthogonal.

1 Last time: least-squares problems

Definition. Suppose A is an $m \times n$ matrix and $b \in \mathbb{R}^m$.

The linear system $A^\top Ax = A^\top b$ is always consistent, so has at least one solution.

A solution to $A^\top Ax = A^\top b$ is called a *least-squares solution* to the equation $Ax = b$.

Let $\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \geq 0$ for $v \in \mathbb{R}^n$. Recall that $\|v\| = 0$ if and only if $v = 0$.

Fact. A vector $s \in \mathbb{R}^n$ is a least-squares solution to $Ax = b$ if and only if $\|b - As\| \leq \|b - Ax\|$ for all x.

The linear system $Ax = b$ is consistent if and only if $\|b - Ax\| = 0$ for some $x \in \mathbb{R}^n$.

This means that if $Ax = b$ is consistent then all least-squares solutions s satisfy $\|b - As\| = 0$ so $As = b$.

If $Ax = b$ is inconsistent, there is still at least one least-squares solution s (but in this case $\|b - As\| > 0$).

Theorem. Let A be an $m \times n$ matrix. The following properties are equivalent:

(a) $Ax = b$ has a unique least-squares solution for each $b \in \mathbb{R}^m$.

(b) The columns of A are linearly independent.

(c) $A^\top A$ is invertible.

Example (Lines of best fit)**.** Suppose we have n data points $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$.

We want to find parameters $\beta_0, \beta_1 \in \mathbb{R}$ such that $y = \beta_0 + \beta_1 x$ describes the *line of best fit* for this data. If our points are all on the same line, then for some $\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \in \mathbb{R}^2$ we would have

$$b_i = \beta_0 + \beta_1 a_i \quad \text{for } i = 1, 2, \ldots, n,$$

meaning that $x = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$ is an exact solution to the linear system $Ax = b$ where

$$A = \begin{bmatrix} 1 & a_1 \\ 1 & a_2 \\ \vdots & \vdots \\ 1 & a_n \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

If the given points are not on the same line, then no exact solution to $Ax = b$ exists, and we should instead try to find a least-squares solution to this linear system.

To be concrete, suppose we have four points $(2, 1), (5, 2), (7, 3)$, and $(8, 3)$ so that

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}.$$

The least-squares solutions to $Ax = b$ are the exact solutions to $A^\top Ax = A^\top b$. We have

$$A^\top A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}$$

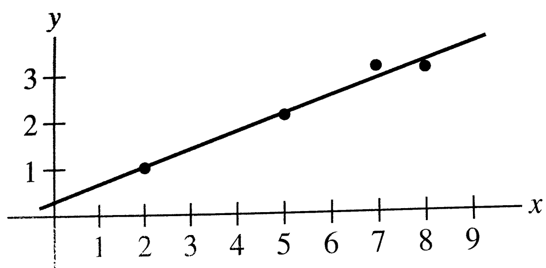
and

$$A^\top b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}.$$

The matrix $A^\top A$ is invertible. (Why?) It follows that a least-squares solution is provided by

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = (A^\top A)^{-1} A^\top b = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 57 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 142 & -22 \\ -22 & 4 \end{bmatrix} \begin{bmatrix} 9 \\ 57 \end{bmatrix} = \begin{bmatrix} 2/7 \\ 5/14 \end{bmatrix}.$$

Thus our line of best fit for the data is $y = \frac{2}{7} + \frac{5}{14}x$:



2 Symmetric matrices

A matrix A is *symmetric* if $A^\top = A$. This happens if A is square and $A_{ij} = A_{ji}$ for all i, j.

Example. $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & 8 \\ 0 & 8 & -7 \end{bmatrix}$ and $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ are symmetric matrices.

$\begin{bmatrix} 1 & -3 \\ 3 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -4 & 0 \\ -6 & 1 & -4 \\ 6 & -6 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$ are not symmetric.

Proposition. If A is a symmetric matrix and k is a positive integer then A^k is also symmetric.

Proof. If $A = A^\top$ then $(A^k)^\top = (AA\cdots A)^\top = A^\top \cdots A^\top A^\top = (A^\top)^k = A^k$. □

Proposition. If A is an invertible symmetric matrix then A^{-1} is also symmetric.

Proof. This is because $(A^{-1})^\top = (A^\top)^{-1}$. □

Recall how we can diagonalize a matrix.

Example. Let $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$.

Then $\det(A - xI) = (8 - x)(6 - x)(3 - x)$ so the eigenvalues of A are 8, 6, and 3. By constructing bases for the null spaces of $A - 8I$, $A - 5I$, and $A - 3I$, we find that the following are eigenvectors of A:

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ with eigenvalue 8.}$$

$v_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ with eigenvalue 6.

$v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with eigenvalue 3.

These eigenvectors are actually an orthogonal basis for $\mathbb{R}^3$.

Converting these vectors to unit vectors gives an orthonormal basis of eigenvectors:

$$u_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \qquad u_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

We then have $A = PDP^{-1}$ where

$$P = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(Why does this hold? It is enough to check that $PDP^{-1}v = Av$ for $v \in \{u_1, u_2, u_3\}$.)

Since the columns of P are orthonormal, we actually have $P^\top = P^{-1}$ so $A = PDP^\top$.

The special properties in this example will turn out to hold for all symmetric matrices.

Theorem. Suppose A is a symmetric matrix. Then any two eigenvectors from different eigenspaces of A are orthogonal. In other words, if $A = A^\top$ is $n \times n$ and $u, v \in \mathbb{R}^n$ are such that $Au = au$ and $Av = bv$ for numbers $a, b \in \mathbb{R}$ with $a \neq b$, then $u \bullet v = 0$.

Proof. Let u and v be eigenvectors of A with eigenvalues a and b, where $a \neq b$.

Then $au \bullet v = Au \bullet v = (Au)^\top v = u^\top A^\top v = u^\top Av = u \bullet Av = u \bullet bv$.

But $au \bullet v = a(u \bullet v)$ and $u \bullet bv = b(u \bullet v)$, so this means $a(u \bullet v) = b(u \bullet v)$ and therefore $(a-b)(u \bullet v) = 0$.

Since $a - b \neq 0$, it follows that $u \bullet v = 0$. □

Recall that a matrix P is *orthogonal* if P is invertible and $P^{-1} = P^\top$.

Definition. A matrix A is *orthogonally diagonalizable* if there is an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1} = PDP^\top$.

When A is orthogonally diagonalizable and $A = PDP^{-1} = PDP^\top$, the diagonal entries of D are the eigenvalues of A, and the columns of P are the corresponding eigenvectors; moreover, these eigenvectors form an orthonormal basis of $\mathbb{R}^n$.

In fact, it follows by the arguments in our earlier lectures about diagonalizable matrices that an $n \times n$ matrix A is orthogonally diagonalizable if and only if there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors for A.

Surprisingly, there is a much more direct characterization of orthogonally diagonalizable matrices:

Theorem. A square matrix is orthogonally diagonalizable if and only if it is symmetric.

We prove this after a sequence of lemmas.

Lemma. If A is orthogonally diagonalizable then A is symmetric.

Proof. If X, Y, Z are $n \times n$ matrices then $(XYZ)^\top = Z^\top (XY)^\top = Z^\top Y^\top X^\top$.

Suppose $A = PDP^\top$ where D is diagonal. Then $D = D^\top$ and $(P^\top)^\top = P$, so

$$A^\top = (PDP^\top)^\top = (P^\top)^\top D^\top P^\top = PDP^\top = A.$$

□

Lemma. All (complex) eigenvalues of an $n \times n$ symmetric matrix A with real entries belong to $\mathbb{R}$.

Proof. Suppose A is a symmetric $n \times n$ matrix with real entries, so that $A = A^\top = \overline{A}$.

Let $v \in \mathbb{C}^n$. Then $\overline{v}^\top Av$ is some complex number.

For example, if $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $v = \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$ then

$$\overline{v}^\top Av = \begin{bmatrix} 1-i & 1+i \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} = \begin{bmatrix} 3+i & 3-i \end{bmatrix} \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} = (3+i)(1+i) + (3-i)(1-i) = 4.$$

In fact, the number $\overline{v}^\top Av$ belongs to $\mathbb{R}$ since $\overline{\overline{v}^\top Av} = v^\top A\overline{v} = \left(\overline{v}^\top Av\right)^\top = \overline{v}^\top Av$.

(The last equality holds since both sides are 1×1 matrices, i.e., scalars.)

Now suppose $v \in \mathbb{C}^n$ is an eigenvector for A with eigenvalue $\lambda \in \mathbb{C}$. Then $\overline{v}^\top Av = \overline{v}^\top(\lambda v) = \lambda(\overline{v}^\top v) \in \mathbb{R}$. The complex number $\overline{v}^\top v$ always belongs to $\mathbb{R}$ (why?) so it must also hold that $\lambda \in \mathbb{R}$. □

Lemma. An $n \times n$ matrix A with all real eigenvalues can be written as $A = URU^\top$ where U is an $n \times n$ orthogonal matrix (i.e., has orthonormal columns) and R is an $n \times n$ upper-triangular matrix.

One calls $A = URU^\top$ with U and R of this form a *Schur factorization* of A.

Proof. Suppose A is an $n \times n$ matrix with all real eigenvalues.

Let $u_1 \in \mathbb{R}^n$ be a unit eigenvector for A with eigenvalue $\lambda \in \mathbb{R}$.

Let $u_2, \dots, u_n \in \mathbb{R}^n$ be any vectors such that $u_1, u_2, \dots, u_n$ is an orthonormal basis for $\mathbb{R}^n$.

(One way to construct these vectors: let $u_1 = x_1, x_2, \dots, x_n$ be any basis, apply the Gram-Schmidt process to get $u_1 = v_1, v_2, \dots, v_n$, and then convert each v_i to a unit vector.)

Define $U = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}$ so that $U^\top = U^{-1}$.

By considering the product $U^\top AUe_i$ for $i = 1, 2, \dots, n$, one finds that $U^\top AU$ has the form

$$U^\top AU = \begin{bmatrix} \lambda & * \\ 0 & B \end{bmatrix}$$

for some $(n-1) \times (n-1)$ matrix B. Here, $*$ stands for $n-1$ arbitrary entries.

The matrix $U^\top AU = U^{-1}AU$ has the same characteristic polynomial as A.

This polynomial is just $(\lambda - x)\det(B - xI)$, which is $\lambda - x$ times the characteristic polynomial of B.

Since the characteristic polynomial of A has all real roots, the same must be true of the characteristic polynomial of B. Thus B must also have all real eigenvalues.

By repeating the argument above, we deduce that there is an eigenvalue $\mu \in \mathbb{R}$ for B, an $(n-1) \times (n-1)$ orthogonal matrix V, and an $(n-2) \times (n-2)$ matrix C with all real eigenvalues such that

$$V^\top BV = \begin{bmatrix} \mu & * \\ 0 & C \end{bmatrix}.$$

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}$ is also orthogonal, and the product of orthogonal matrices is orthogonal. (Why?)

It follows for the orthogonal matrix $W = U \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}$ that $W^\top AW = \begin{bmatrix} \lambda & * & * \\ 0 & \mu & * \\ 0 & 0 & C \end{bmatrix}$.

By continuing in this way, we will eventually construct an orthogonal matrix X and an upper-triangular matrix R such that $X^\top AX = R$, in which case $A = XX^\top AXX^\top = XRX^\top$. □

Now we can prove the theorem.

Proof of theorem. The first lemma shows that if A is orthogonally diagonalizable then A is symmetric.

Suppose conversely that A is symmetric. Then A has all real eigenvalues, so there exists a Schur factorization $A = URU^\top$. We then have $A^\top = (URU^\top)^\top = UR^\top U^\top$ but also $A^\top = A = URU^\top$.

Since $U^\top = U^{-1}$, it follows that $R = R^\top$. Since R is upper-triangular, this can only hold if R is diagonal.

But if R is diagonal then $A = URU^\top$ is orthogonally diagonalizable. □

To orthogonally diagonalize an $n \times n$ symmetric matrix A, we just need to find an orthogonal basis of eigenvectors $v_1, v_2, \ldots, v_n$ for $\mathbb{R}^n$. Then $A = UDU^\top$ with $U = \begin{bmatrix} u_1 & u_2 & \ldots & u_n \end{bmatrix}$ where $u_i = \frac{1}{\|v_i\|} v_i$ and D is the diagonal matrix of the corresponding eigenvalues.

If all eigenspaces of A are 1-dimensional, then any basis of eigenvectors will be orthogonal. If A has an eigenspace of dimension greater than one, then after finding a basis for this eigenspace, it is necessary to apply the Gram-Schmidt process to convert this basis to one that is orthogonal.

Corollary. If $A = UDU^\top$ where $U = \begin{bmatrix} u_1 & u_2 & \ldots & u_n \end{bmatrix}$ has orthonormal columns and

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

is diagonal, then $A = \lambda_1 u_1 u_1^\top + \lambda_2 u_2 u_2^\top + \cdots + \lambda_n u_n u_n^\top$.

Each product $u_i u_i^\top$ is an $n \times n$ matrix of rank 1. One calls this expression a *spectral decomposition* of A.

Example. Let $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$. A spectral decomposition of A is given by

$$A = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$= 8 \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} + 3 \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 32/5 & 16/5 \\ 16/5 & 8/5 \end{bmatrix} + \begin{bmatrix} 3/5 & -6/5 \\ -6/5 & 12/5 \end{bmatrix}.$$

3 Vocabulary

Keywords from today's lecture:

1. **Symmetric matrix**.

   A matrix A that is equal to its transpose, so that $A = A^\top$. Such a matrix is square.

   Symmetric matrices are precisely the square matrices A that are **orthogonally diagonalizable**, in other words, the matrices that can be expressed as

   $$A = PDP^\top$$

   where D is a diagonal matrix and P is an invertible matrix with $P^{-1} = P^\top$.

   Example: $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ or any diagonal matrix.

2. **Schur factorization** of an $n \times n$ matrix A.

   A decomposition $A = URU^\top$ where R is an $n \times n$ upper triangular matrix and U is an orthogonal matrix (i.e., U is invertible with $U^{-1} = U^\top$).