Instructions: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions (either because of difficulty, being open-ended, or requiring external resources) are marked with a star. These problems may still offer useful practice with the core concepts in the course.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit. **Please handwrite your answers and show all steps in your calculations**, as you would on an exam.

To get full credit for the offline homework, you just need to make a good-faith attempt on the required problems. The bar for receiving extra credit points is higher.

Show all steps and provide justification for all answers.

1. Suppose $V = \{p(x) = c_0 + c_1x + c_2x^2 + c_3x^3 : c_0, c_1, c_2, c_3 \in \mathbb{R}\}$ is the 4-dimensional vector space of polynomials of degree ≤ 3. Let $T : V \to V$ be the linear map defined by $T(p(x)) = p(1-x)$.

   This means that $T(1 + 2x + x^3) = 1 + 2(1-x) + (1-x)^3 = 4 - 5x + 3x^2 - x^3$, for example.

   Let $a_i = x^{i-1}$ and $b_i = (x+1)^{i-1}$ for $i = 1, 2, 3, 4$.

   Then a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are two bases for V.

   As usual let e_1, e_2, e_3, e_4 be the standard basis of $\mathbb{R}^4$.

   There are invertible linear maps $f, g : \mathbb{R}^4 \to V$ with $f(e_i) = a_i$ and $g(e_i) = b_i$ for all $i = 1, 2, 3, 4$.

   This means that $f^{-1} \circ T \circ f$ and $g^{-1} \circ T \circ g$ and $f^{-1} \circ g$ are all linear maps $\mathbb{R}^4 \to \mathbb{R}^4$.

   Let A, B, and P be the standard matrices of $f^{-1} \circ T \circ f$, $g^{-1} \circ T \circ g$, and $f^{-1} \circ g$, respectively.

   Compute these matrices, and check that $\det(A) = \det(B)$ and $\mathrm{tr}(A) = \mathrm{tr}(B)$.

*2. Suppose V is an n-dimensional vector space and $T : V \to V$ is a linear map.

   Assume $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are two bases for V.

   As usual let $e_1, e_2, \dots, e_n$ be the standard basis of $\mathbb{R}^n$.

   There are invertible linear maps $f, g : \mathbb{R}^n \to V$ with $f(e_i) = a_i$ and $g(e_i) = b_i$ for all $i = 1, 2, \dots, n$.

   This means that $f^{-1} \circ T \circ f$ and $g^{-1} \circ T \circ g$ and $f^{-1} \circ g$ are all linear maps $\mathbb{R}^n \to \mathbb{R}^n$.

   Let A, B, and P be the standard matrices of $f^{-1} \circ T \circ f$, $g^{-1} \circ T \circ g$, and $f^{-1} \circ g$, respectively.

   Find an expression for A in terms of B and P, and use this to explain why A and B have the same trace and the same determinant.

   The determinant and trace of T are **defined** to be the values of $\det(A) = \det(B)$ and $\mathrm{tr}(A) = \mathrm{tr}(B)$. To compute these values, you have to pick a basis for V, but this exercise shows that the numbers you get are the same no matter which basis you use.

3. Adopt the same setup as in the previous problem.

   Explain why it still holds that T is invertible if and only $\det(T) \neq 0$.

   In other words, explain why T is an invertible linear map if and only if the matrix A is invertible.

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

*4. Suppose $v_1, v_2 \in \mathbb{R}^2$ are nonzero vectors. This exercise walks through a proof of the sometimes useful identity $v_1 \bullet v_2 = \|v_1\|\|v_2\|\cos(\theta)$ where θ is the angle between v_1 and v_2.

To define the angle θ precisely, assume that rotating v_1 counterclockwise by θ radians gives a **positive** scalar multiple of v_2, and that v_1 and v_2 are labeled such that this angle has $\theta \in [0, \pi]$. (This means that if we have to go more than π radians counterclockwise from v_1 to get to v_2, then we switch the names of the vectors.)

(a) Define u_1 and u_2 to be the unit vectors in the directions of v_1 and v_2.

Explain why $v_1 \bullet v_2 = \|v_1\|\|v_2\|(u_1 \bullet u_2)$.

(b) Suppose $\psi \in [0, 2\pi)$ and $M = \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix}$. Check that $(Mu_1) \bullet (Mu_2) = u_1 \bullet u_2$.

(c) Explain why you can choose a value of ψ such that $Mu_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $Mu_2 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ where θ is the angle between v_1 and v_2. Conclude that $u_1 \bullet u_2 = \cos(\theta)$.

Combining all three parts tells us that $v_1 \bullet v_2 = \|v_1\|\|v_2\|\cos(\theta)$.

5. Do there exist two linearly independent vectors in $\mathbb{R}^4$ that are orthogonal to all three of the vectors

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 7 \\ 1 \\ 4 \\ 7 \end{bmatrix}?$$

Find two such vectors if they exist, and otherwise explain why there are no such vectors.

6. Consider the plane $P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 3x - 2y + 6z = 0 \right\}$ in $\mathbb{R}^3$.

(a) The subspace P is 2-dimensional. Find an orthogonal basis for P.

(b) Find the vector in P that is closest to $v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

7. Find an orthonormal basis for the subspace of vectors of the form

$$\begin{bmatrix} a + 3b + 2c \\ 3a + 2b + 4c \\ 2a + 5b + 4c \\ 6a + 5b + 4c \end{bmatrix}$$

where $a, b, c \in \mathbb{R}$ are real numbers.

*8. Suppose A is a square matrix with all entries in $\mathbb{R}$.

Let $a, b \in \mathbb{R}$ and suppose $\lambda = a + bi \in \mathbb{C}$ is an eigenvalue for A.

(a) Show that if $A^\top = -A$ then $a = 0$.

(b) Show that if $A^\top = A$ then $b = 0$.

(c) Show that if $A^\top = A^{-1}$ then $a^2 + b^2 = 1$.

*9. Suppose A is a square matrix with all entries in $\mathbb{C}$. Determine whether the eigenvalue properties in the previous exercise still hold. That is, for each of the three statements, either find a counterexample or show that the given property for λ is still true when A has complex entries.

*10. Define

$$\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad j = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad k = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

As with complex numbers, when $a, b, c, d \in \mathbb{R}$ we abbreviate by setting

$$a + bi + cj + dk = a\mathbf{1} + bi + cj + dk = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}.$$

Now consider the real vector space of *quaternionic numbers* $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$.

(a) Compute all nine products yz for $y, z \in \{i, j, k\}$.

Conclude that $\mathbb{H}$ is closed under multiplication, that is, if $y, z \in \mathbb{H}$ then $yz \in \mathbb{H}$.

However, multiplication in $\mathbb{H}$ is not commutative (as it is in $\mathbb{R}$ and $\mathbb{C}$).

Why is multiplication in $\mathbb{H}$ associative? (In the sense that $x(yz) = (xy)z$ for all $x, y, z \in \mathbb{H}$.)

(b) Suppose $z = a + bi + cj + dk \in \mathbb{H}$. Find a formula for $\det(z)$.

(c) Suppose $z = a + bi + cj + dk \in \mathbb{H}$ is nonzero.

Find a formula for z^{-1} and check that this element is also still in $\mathbb{H}$.