Instructions: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions (either because of difficulty, being open-ended, or requiring external resources) are marked with a star. These problems may still offer useful practice with the core concepts in the course.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit. **Please handwrite your answers and show all steps in your calculations**, as you would on an exam.

To get full credit for the offline homework, you just need to make a good-faith attempt on the required problems. The bar for receiving extra credit points is higher.

Show all steps and provide justification for all answers.

1. Find an orthogonal matrix U and a diagonal matrix D such that

$$A = UDU^T \quad \text{for the matrix } A = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 8 & 8 \\ 4 & 8 & 8 \end{bmatrix}.$$

2. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

   Show all steps in your derivation by hand to receive credit (but feel free to check your answer with a calculator).

3. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

   Show all steps in your derivation by hand to receive credit (but feel free to check your answer with a calculator).

4. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

   Show all steps in your derivation by hand to receive credit (but feel free to check your answer with a calculator).

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

5. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Show all steps in your derivation by hand to receive credit (but feel free to check your answer with a calculator).

6. Suppose A is a 2×2 matrix with a singular value decomposition

$$A = U\Sigma V^T$$

where U and V are orthogonal 2×2 matrices and

$$\Sigma = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}.$$

The first column of U is the vector $\begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$.

Draw a picture of the region in $\mathbb{R}^2$ given by

$$\left\{ Ax : x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \text{ is a vector with } x_1^2 + x_2^2 \leq 1 \right\}.$$

7. Suppose $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ are orthonormal vectors. Assume $k < n$.

   (a) Describe an algorithm to find $n - k$ vectors

   $$v_{k+1}, v_{k+2}, \dots, v_n \in \mathbb{R}^n$$

   such that $v_1, v_2, \dots, v_n$ is an orthonormal basis for $\mathbb{R}^n$.

   (b) Suppose $n = 3$ and $v_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$.

   Find two vectors $v_2, v_3 \in \mathbb{R}^3$ such that v_1, v_2, v_3 is an orthonormal basis for $\mathbb{R}^n$.

*8. Suppose A is an $m \times n$ matrix with at most one nonzero entry in each row and column.

Describe a singular value decomposition for A.

Use the following notation in your answer: suppose the positions of A with nonzero entries are

$$(i_1, j_1), (i_2, j_2), \dots, (i_r, j_r)$$

where $j_1 < j_2 < \cdots < j_r$, and the entries in these positions are $a_1, a_2, \dots, a_r \in \mathbb{R}$.

(It may be useful to compare your answer with #2 and #5.)

*9. Suppose A is an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$.

Suppose σ_k is much larger than σ_{k+1} where $k < \text{rank}(A)$.

Describe an algorithm using an SVD for A that produces a good rank k approximation to A.

Apply this algorithm (using a computer or calculator) to find a rank 2 approximation to the matrix

$$A = \begin{bmatrix} 0.2 & 0.1 & -0.1 \\ 1.2 & 0.1 & 0.8 \\ 1.0 & -2.0 & 5.5 \end{bmatrix}.$$

*10. Suppose A is an invertible 3×3 matrix with a singular value decomposition

$$A = \left[\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\right] \left[\begin{array}{ccc} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{array}\right] \left[\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}\right]^\top.$$

Give a geometric interpretation of the numbers $\sigma_1, \sigma_2, \sigma_3$ and the vectors $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}^3$.

Your answer should involve the sphere $\mathbb{S} = \{w \in \mathbb{R}^3 : \|w\| \leq 1\}$ and its image $A\mathbb{S} = \{Aw : w \in \mathbb{S}\}$.

11. Use the rank-nullity theorem and the dimension formula for the orthogonal complement to show that if A is any $m \times n$ matrix then $\mathrm{rank}(A) = \mathrm{rank}(A^\top)$.

12. Let A be a square $n \times n$ matrix whose rows are orthonormal. Prove that the columns of A are orthonormal.

*13. Let A be an $m \times n$ matrix. Describe an algorithm to find a unit vector $v \in \mathbb{R}^n$ such that

$$\|Av\| = \max\{\|Ax\| : x \in \mathbb{R}^n \text{ with } \|x\| = 1\}.$$