Instructions: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions (either because of difficulty, being open-ended, or requiring external resources) are marked with a star. These problems may still offer useful practice with the core concepts in the course.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit. **Please handwrite your answers and show all steps in your calculations**, as you would on an exam.

To get full credit for the offline homework, you just need to make a good-faith attempt on the required problems. The bar for receiving extra credit points is higher.

1. Warmup: find general formulas for the solutions of the linear systems

$$\begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + 4x_2 = 3 \end{cases} \qquad \text{and} \qquad \begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + 6x_2 = 3. \end{cases}$$

   Now suppose a, b, c, d, p, q are real numbers with $ad - bc \neq 0$.

   Find a general formula for the solution of the linear system $\begin{cases} ax_1 + bx_2 = p \\ cx_1 + dx_2 = q. \end{cases}$

*2. Suppose n is a positive integer and $a_1, a_2, \dots, a_n$ are real numbers.

   What condition must be satisfied for the linear system

$$\begin{cases} x_1 - x_2 = a_1 \\ x_2 - x_3 = a_2 \\ x_3 - x_4 = a_3 \\ \vdots \\ x_{n-1} - x_n = a_{n-1} \\ x_n - x_1 = a_n \end{cases}$$

   to be consistent? Write down the augmented matrix of this linear system. Assuming the system is consistent, find a general formula for its solutions.

3. Suppose we have two linear systems with the same number of equations and the same number of variables. Then the systems' augmented matrices have the same size. If the augmented matrices are row equivalent then the systems are equivalent, meaning they have the same solutions. Do there exist two linear systems, both with m equations and n variables, that are equivalent but whose augmented matrices are **not** row equivalent? Explain why this is impossible or find an example.

4. There is a way to multiply two 2×2 matrices to get another 2×2 matrix. The formula is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}.$$

   (a) Find a 2×2 matrix I with $\begin{bmatrix} a & b \\ c & d \end{bmatrix} I = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for all $a, b, c, d \in \mathbb{R}$. What is $I \begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

(b) Find a 2×2 matrix E with $E\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} y & z \\ w & x \end{bmatrix}$ for all $w,x,y,z \in \mathbb{R}$. What is $\begin{bmatrix} w & x \\ y & z \end{bmatrix} E$?

(c) Find a 2×2 matrix J with $J^2 = -I$ where we define $-\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$.

The matrices you find for this question should have all real entries.

*5. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix with real entries such that $ad - bc \neq 0 \neq b$.

Show that there are numbers $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{R}$ with

$$A = \begin{bmatrix} p_1 & 0 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} q_1 & 0 \\ q_2 & q_3 \end{bmatrix}.$$

6. If A is a 1×3 matrix then $\mathsf{RREF}(A)$ either has the form

$$\begin{bmatrix} 1 & * & * \end{bmatrix} \quad\text{or}\quad \begin{bmatrix} 0 & 1 & * \end{bmatrix} \quad\text{or}\quad \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad\text{or}\quad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

where each $*$ means an arbitrary real number. In the first, second, and fourth cases, A is the augmented matrix of a linear system in two variables with infinitely many solutions. In the third case, A is the augmented matrix of a linear system in two variables with zero solutions.

Describe with similar notation what the possibilities are for $\mathsf{RREF}(A)$ if A is a 2×3 matrix. In each case, indicate how many solutions there are for the linear system whose augmented matrix is A.

7. What are the possibilities for $\mathsf{RREF}\left(\begin{bmatrix} x & 1 \\ y & 2 \end{bmatrix}\right)$ if x and y are arbitrary real numbers? Draw a picture of the xy-plane in which you identify the regions of points (x, y) where $\mathsf{RREF}\left(\begin{bmatrix} x & 1 \\ y & 2 \end{bmatrix}\right)$ takes its different possible values.

8. If the reduced echelon form of the augmented matrix of some linear system is

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

then what is the general formula for the solution to the linear system?

Going in the opposite direction, find a matrix A that is the augmented matrix of a linear system with 3 equations and 4 variables whose general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - 3a - 4b \\ a \\ 2 - 5b \\ b \end{bmatrix} \text{ for all } a, b \in \mathbb{R}.$$

9. Arrange the first 9 digits of your student ID as the entries of a 3×3 matrix A. (If you run out of digits, reuse the first few digits.) Compute $\mathsf{RREF}(A)$ by hand without using a calculator, showing all of your work in the intermediate steps of the row reduction algorithm.

*10. A *linear inequality* is an equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b$ where $a_1, a_2, \ldots, a_m, b \in \mathbb{R}$ are numbers and $x_1, x_2, \ldots, x_n$ are variables. A solution to a linear inequality is an assignment of numbers to the variables which makes the inequality true. Systems of linear inequalities and their solutions are defined similarly.

Make a picture of the lines $x_1 + x_2 = 0$ and $x_1 - x_2 = 4$ and $2x_1 + x_2 = 6$ and compute the three points where two of the lines intersect. These lines divide the $\mathbb{R}^2$ plane into 7 regions. Explain why each of these regions is the set of solutions to a system of linear inequalities in the variables x_1 and x_2. Identify the corresponding system for each region.

*11. A *quadratic equation* in two variables x_1, x_2 is an equation of the form

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 = f$$

for real numbers a, b, c, d, e, f. A *quadratic system* is a list of quadratic equations.

While the set of solutions (x_1, x_2) to a linear equation (in two variables) forms a line in the Cartesian plane, the set of solutions to a quadratic equation (in two variables) forms a **conic section** in the Cartesian plane. A conic section is a curve given by an **ellipse**, **parabola**, or **hyperbola**. A circle is a special case of an ellipse and a straight line is considered to be a degenerate case of a parabolic. See `https://en.wikipedia.org/wiki/Conic_section` for more information about what these shapes look like.

What are the possibilities for the number of solutions to a quadratic system in two variables? (In this problem, a **solution** means a real-valued solution (x_1, x_2) with $x_1, x_2 \in \mathbb{R}$.) Justify your answer by adapting the geometric argument in Lecture 1 that was used to prove that every linear system in two variables has 0, 1, or infinitely many solutions. Draw a picture corresponding to each different possibility for the number of solutions.