Instructions: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions (either because of difficulty, being open-ended, or requiring external resources) are marked with a star. These problems may still offer useful practice with the core concepts in the course.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit. **Please handwrite your answers and show all steps in your calculations**, as you would on an exam.

To get full credit for the offline homework, you just need to make a good-faith attempt on the required problems. The bar for receiving extra credit points is higher.

1. Assume h and k are real numbers and consider the linear system
$$\begin{aligned} x_1 + 4x_2 &= k \\ 3x_1 + hx_2 &= 7. \end{aligned}$$
Determine all values of h and k such that this system has (i) zero solutions, (ii) a unique solution, or (iii) infinitely many solutions.

2. Suppose $v_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ -6 \\ -9 \end{bmatrix}$, and $w = \begin{bmatrix} 6 \\ -10 \\ a \end{bmatrix}$ where $a \in \mathbb{R}$.
Find all values of a such that w is in $\mathbb{R}$-span$\{v_1, v_2, v_3\}$.

3. Describe all matrices A such that $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$ and $A\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$.

4. Assume A is a 3×3 matrix. Only the first and second columns of A are pivot columns and
$$A\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
What is $\mathsf{RREF}(A)$? What is the general solution to $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$?

5. Find all values of $h \in \mathbb{R}$ for which the vectors
$$\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} -5 \\ 5 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 7 \\ h \end{bmatrix}$$
are linearly dependent.

6. Suppose $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ are linearly independent vectors.
   (a) If we add another vector v_{k+1} to this list, will it always still be linearly independent?

   What can you say about when the larger list of vectors is still linearly independent?

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

(b) If we delete one of the vectors from the list, say v_k, will it always still be linearly independent?

What can you say about when the smaller list of vectors is still linearly independent?

Explain and justify your answers to both parts.

7. Choose m vectors $v_1, v_2, \dots, v_m \in \mathbb{R}^n$.

   Suppose $\mathbb{R}$-span$\{v_1, v_2, \dots, v_m\} = \mathbb{R}^n$. A theorem in Lecture 4 states that this occurs if and only the matrix $A = \left[\begin{array}{cccc} v_1 & v_2 & \dots & v_m \end{array}\right]$ has a pivot position in every row.

   Use this fact to explain why we must have $m \geq n$.

*8. Let $V = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_n \end{array}\right] \in \mathbb{R}^n : a_1 + a_2 + \dots + a_n = 0 \right\}$.

   Find a set of $n-1$ vectors in $\mathbb{R}^n$ with span equal to V.

   Then use the previous exercise to show that no set of $n-2$ vectors in $\mathbb{R}^n$ has span equal to V.

   Hint: show if we have a set spanning V then we can add 1 vector to get a set spanning $\mathbb{R}^n$.

*9. Choose an angle $\theta \in [0, 2\pi)$.

   What is the relationship between the vector $\left[\begin{array}{c} v_1 \\ v_2 \end{array}\right] \in \mathbb{R}^2$ and $\left[\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array}\right] \left[\begin{array}{c} v_1 \\ v_2 \end{array}\right]$?

   Draw a picture and justify your answer.

   Then use your answer to prove the trigonometric sum-angle formulas

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) \quad \text{and} \quad \sin(\theta + \phi) = \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi).$$

   Hint: answer the question for $\left[\begin{array}{c} v_1 \\ v_2 \end{array}\right] = \left[\begin{array}{c} v_1 \\ 0 \end{array}\right]$ *and* $\left[\begin{array}{c} v_1 \\ v_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ v_2 \end{array}\right]$, *using the unit circle definitions of* sin *and* cos. *Then use parallelogram rule for vector addition to deduce the general answer.*

10. Suppose a and b are real numbers. Consider the lines

$$\mathcal{L}_1 = \left\{ \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] \in \mathbb{R}^2 : x_2 = ax_1 \right\} \quad \text{and} \quad \mathcal{L}_2 = \left\{ \left[\begin{array}{c} y_1 \\ y_2 \end{array}\right] \in \mathbb{R}^2 : y_2 = by_1 \right\}.$$

   Find all values of a and b such that there is exactly one way of writing

$$\left[\begin{array}{c} 2023 \\ 2121 \end{array}\right] = v + w$$

   with $v \in \mathcal{L}_1$ and $w \in \mathcal{L}_2$. Find a formula for v and w in this case.

*11. Suppose $v, w \in \mathbb{R}^n$ are vectors with $v_i w_j - v_j w_i = 0$ for all $1 \leq i < j \leq n$. Show that v and w are linearly dependent, that is, one vector is a scalar multiple of the other.

*12. There are $n \geq 3$ different particles $p_1, p_2, \dots, p_n$ arranged in a circle. If $1 < i < n$ then p_i is between particles p_{i-1} and p_{i+1}, while p_1 is between p_n and p_2 (and p_n is between p_{n-1} and p_1). The temperature of each particle is the average of the temperatures of its two adjacent neighbors. Must every particle have the same temperature? Justify your answer.