Instructions: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions (either because of difficulty, being open-ended, or requiring external resources) are marked with a star. These problems may still offer useful practice with the core concepts in the course.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit. **Please handwrite your answers and show all steps in your calculations**, as you would on an exam.

To get full credit for the offline homework, you just need to make a good-faith attempt on the required problems. The bar for receiving extra credit points is higher.

Show all steps and provide justification for all answers.

1. Let $A = \begin{bmatrix} 1 & 1 & 1 & 11 \\ 6 & 8 & 16 & -26 \\ 3 & 1 & -7 & 25 \end{bmatrix}$.
   (a) Find a basis for the column space of A.
   (b) Find a basis for the null space of A.

   Do all steps by hand and show your work.
2. Let $A = \begin{bmatrix} 2 & 3 & 16 & -5 \\ -6 & 8 & 20 & -2 \\ 3 & -2 & -8 & 3 \end{bmatrix}$.
   (a) Find a basis for the column space of A.
   (b) Find a basis for the null space of A.

   Do all steps by hand and show your work.
3. Let $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$ be a one-row matrix. Here $a_1, a_2, \dots, a_n \in \mathbb{R}$ are arbitrary numbers.
   (a) Find a basis for the column space of A.
   (b) Find a basis for the null space of A.

   Your answer will depend on the entries in A. Remember to the consider the case when $A = 0$.
4. Suppose $v_1, v_2, \dots, v_k$ are linearly independent vectors in $\mathbb{R}^n$, where $k < n$.

   Describe an algorithm to find vectors $v_{k+1}, v_{k+2}, \dots, v_n$ such that $v_1, v_2, \dots, v_n$ is a basis for $\mathbb{R}^n$.

*5. Show that the rank one $m \times n$ matrices (the $m \times n$ matrices A with $\mathrm{rank}(A) = \dim(\mathrm{Col}(A)) = 1$) are precisely the matrices that can be expressed as vw^T for vectors $0 \neq v \in \mathbb{R}^m$ and $0 \neq w \in \mathbb{R}^n$.

   One way to do this is by the following steps:
   (a) Explain why $\mathrm{rank}(vw^T) = 1$ if $0 \neq v \in \mathbb{R}^m$ and $0 \neq w \in \mathbb{R}^n$.
   (b) Explain why a rank one matrix A must have a nonzero column.
   (c) If v is a nonzero column of a rank one matrix A, explain how to find a nonzero vector w such that $A = vw^T$.

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

6. Let $A = \begin{bmatrix} 2 & 0 & 4 & 0 \\ 1 & 8 & 0 & 5 \\ 2 & -8 & 4 & -4 \\ 0 & 0 & x & -9 \end{bmatrix}$.

   Determine the values of x such that A is invertible and find a formula for A^{-1} in this case.

*7. Suppose A is an $m \times n$ matrix and I_m is the $m \times m$ identity matrix.

   (a) Explain why there exists a matrix B with $AB = I_m$ if A has a pivot position in every row.

   (b) Prove that A has a pivot position in every row if there exists a matrix B such that $AB = I_m$

*8. Suppose A is an $m \times n$ matrix and I_n is the $n \times n$ identity matrix.

   (a) Explain why there exists a matrix B with $BA = I_n$ if A has a pivot position in every column.

   (b) Prove that A has a pivot position in every column if there exists a matrix B such that $BA = I_n$.

9. Suppose A is an $m \times n$ matrix and B is an $n \times q$ matrix. If $\mathrm{rank}(A) = n$ and $\mathrm{rank}(B) = r$, then what is $\mathrm{rank}(AB)$ in terms of m, n, q, and r? Justify your answer.

10. Let $v = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and $w \in \mathbb{R}^3$. Suppose there exists a 3×3 matrix A whose null space **and** column space contains both v and w. What are the possibilities for w? For each of these possibilities, give an example of a 3×3 matrix A whose null space and column space contains both v and w.

**11. Let $\mathbb{R}^\infty$ be the set of vectors $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{bmatrix}$ with an infinite number of rows and all entries $v_i \in \mathbb{R}$.

   Vector addition and scalar multiplication for elements of $\mathbb{R}^\infty$ are defined as you would expect.

   The *span* of a set of vectors in $\mathbb{R}^\infty$ is the set of all of their *finite* linear combinations: even if your set has infinitely many elements, a linear combination can only involve finitely many of them (because there is no operation to add together an infinite set of nonzero vectors).

   For example, we can define the vectors $e_i \in \mathbb{R}^\infty$ as usual to have a 1 is row i and 0 in all other rows. The span of $e_1, e_2, e_3, \dots$ is *not* all of $\mathbb{R}^\infty$ because it does not contain the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$.

   Warmup: what subspace is the span of $e_1, e_2, e_3, \dots$ in $\mathbb{R}^\infty$? Then prove that if $b_1, b_2, b_3, \dots \in \mathbb{R}^\infty$ is any sequence of vectors indexed by the positive integers, then $\mathbb{R}\text{-span}\{b_1, b_2, b_3, \dots\} \neq \mathbb{R}^\infty$.

12. Let A be an $n \times n$ matrix. The *generalized column space* of A is the set of vectors $v \in \mathbb{R}^n$ that are in $\mathrm{Col}(A^k)$ for **every** $k \in \{1, 2, 3, \dots\}$. Denote this subset by $\mathrm{Col}^+ A$.

   Similarly, the *generalized null space* of A is the set of vectors $v \in \mathbb{R}^n$ that are in $\mathrm{Nul}(A^k)$ for **at least one** $k \in \{1, 2, 3, \dots\}$. Denote this subset by $\mathrm{Nul}^+ A$.

   Show that $\mathrm{Nul}^+ A$ and $\mathrm{Col}^+ A$ are both subspaces of $\mathbb{R}^n$.

*13. Show that $v \mapsto Av$ defines an invertible function $\mathrm{Col}^+ A \to \mathrm{Col}^+ A$. Using this fact or some other method, show that the subspaces $\mathrm{Nul}^+ A$ and $\mathrm{Col}^+ A$ are **disjoint** in the sense that only the zero vector is in both $\mathrm{Nul}^+ A$ and $\mathrm{Col}^+ A$.

*14. Show that for each $u \in \mathbb{R}^n$ there are **unique** vectors $v \in \mathrm{Col}^+ A$ and $w \in \mathrm{Nul}^+ A$ with $u = v + w$.

   (Use the previous exercise.)