Instructions: *Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.*

Due on **Wednesday, March 8**.

Unless mentioned otherwise, any Lie algebras below are finite-dimensional and defined over an algebraically closed field $\mathbb{F}$ of characteristic zero.

1. Let $L = L_1 \oplus L_2 \oplus \cdots \oplus L_m$ be the decomposition of a semisimple Lie algebra L into its simple ideals. Fix $X \in L$ and let $X_i \in L_i$ be the unique elements such that $X = X_1 + X_2 + \cdots + X_m$. Show that the semisimple and nilpotent parts of X are the sums of the semisimple and nilpotent parts of the various components X_i; that is, show that $X_s = \sum_i (X_i)_s$ and $X_n = \sum_i (X_i)_n$.
2. Using the standard basis for $L = \mathfrak{sl}_2(\mathbb{F})$, write down the Casimir element of the adjoint representation of L.
3. Prove that if L is a solvable Lie algebra then every irreducible representation of L is one-dimensional.
4. Use Weyl's theorem to prove that $\operatorname{ad} L = \operatorname{Der} L$ when L is semisimple. Recall that $\operatorname{Der} L$ is defined to be the set of linear maps $\delta : L \to L$ satisfying $\delta([X,Y]) = [X, \delta(Y)] + [\delta(X), Y]$ for $X, Y \in L$.
5. A Lie algebra L is *reductive* if $\operatorname{Rad} L = Z(L)$. Suppose L is reductive. Show that L is a completely reducible $\operatorname{ad} L$-module, that $L = Z(L) \oplus [L, L]$ (as Lie algebras), and that $[L, L]$ is semisimple.
6. Show that the classical Lie algebras of types A_n, B_n, C_n, and D_n are semisimple. (Derive this as simply as possible — if your argument involves a lot of tedious calculations, then feel free to skip writing down all the details and just explain what needs to be computed.)
7. Let L be a simple Lie algebra. Suppose β and γ are two bilinear forms $L \times L \to \mathbb{F}$ that are nondegenerate, symmetric, and associative. Use Schur's Lemma to prove that β is a nonzero scalar multiple of γ.
8. Suppose L is a Lie algebra and M is a finite-dimensional L-module. Describe a basis for the L-module $(M \otimes M)^*$ and explain what the action of L is on these basis elements. Identify a natural isomorphism from $(M \otimes M)^*$ to the vector space of bilinear forms $\beta : M \times M \to \mathbb{F}$. Show that if $M = L$ (viewed as an L-module by the formula $X \cdot Y = \operatorname{ad}(X)(Y) = [X, Y]$) and β is identified with an element of $(L \otimes L)^*$ via this isomorphism, then β is associative if and only if $L \cdot \beta = 0$.