Instructions: *Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.*

This assignment is a little longer than usual since the due date is after spring break.

Due on **Wednesday, April 12**.

Except when mentioned otherwise, all Lie algebras and vector spaces below are finite-dimensional and defined over an algebraically closed field $\mathbb{F}$ of characteristic zero.

1. Compute the basis of $\mathfrak{sl}_n(\mathbb{F})$ which is dual to the standard basis via the Killing form κ.

   (This is an upgrade of an earlier exercise which considered the case $n = 2$.)

   Use this to compute t_α and $h_\alpha := 2t_\alpha/\kappa(t_\alpha, t_\alpha)$ for each $\alpha \in \Phi$, relative to the root space decomposition in which $H \subset \mathfrak{sl}_n(\mathbb{F})$ is the maximal toral subalgebra of traceless diagonal matrices.

   Recall that $t_\alpha \in H$ for $\alpha \in H^*$ is the unique element with $\kappa(t_\alpha, h) = \alpha(h)$ for all $h \in H$.

2. Let L be a semisimple Lie algebra with a maximal toral subalgebra H.

   Prove that if $h \in H$ then the centralizer $C_L(h) := \{X \in L : [X, h] = 0\}$ is a reductive Lie algebra.

   Prove that there are elements $h \in H$ with $C_L(h) = H$.

   For which $h \in H$ does this hold if $L = \mathfrak{sl}_n(\mathbb{F})$ and H is the subalgebra of traceless diagonal matrices?

If $\Phi \subset E$ and $\Phi' \subset E'$ are root systems, then an isomorphism $\Phi \to \Phi'$ is a linear bijection $f : E \to E'$ with $f(\Phi) = \Phi'$ and $\langle f(\beta), f(\alpha)\rangle = \langle \beta, \alpha\rangle$ for all $\alpha, \beta \in \Phi$, where $\langle \beta, \alpha\rangle = 2(\beta, \alpha)/(\alpha, \alpha)$. The map f does not need to be an isometry with respect to the forms $(\cdot, \cdot)$ on E and E'.

3. Prove that every three dimensional semisimple Lie algebra has the same root system as $\mathfrak{sl}_2(\mathbb{F})$ up to isomorphism, and so is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$.

4. Prove that no four, five, or seven dimensional semisimple Lie algebras exist.

Below, let Φ be a root system with ambient space E, associated form $(\cdot, \cdot)$, and Weyl group W.

5. Define $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

   Check that $\langle \alpha^\vee, \beta^\vee\rangle = \langle \beta, \alpha\rangle$ for $\alpha, \beta \in \Phi$.

   Prove that $\Phi^\vee$ is a root system (in the same vector space) whose Weyl group is isomorphic to W.

   Draw a picture of $\Phi^\vee$ versus Φ when Φ has type A_1, A_2, B_2, and G_2.

6. Let Φ' be a nonempty subset of Φ with $-\Phi' = \Phi'$, such that if $\alpha, \beta \in \Phi'$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi'$. Prove that Φ' is a root system in the subspace of E that it spans.

7. An *automorphism* of Φ is an isomorphism of root systems from Φ to itself.

   Prove that W is a normal subgroup of the group $\mathrm{Aut}(\Phi)$ of all automorphisms of Φ.

8. Show by example that it can happen that $\alpha - \beta \in \Phi$ for $\alpha, \beta \in \Phi$ with $(\alpha, \beta) \leq 0$.

9. For $\gamma \in E$ let $P_\gamma = \{v \in E : (v, \gamma) > 0\}$. Prove for any finite set of linearly independent vectors $\gamma_1, \gamma_2, \ldots, \gamma_k \in E$ that the intersection $\bigcap_{i=1}^k P_{\gamma_i}$ is nonempty.

10. Fix a simple system Δ for Φ and define $\Phi^\pm \subset \Phi$ accordingly.

    Prove that there is a unique element $w_0 \in W$ with $w_0(\Phi^+) = \Phi^-$.

    Prove that any reduced expression for w_0 must involve every simple reflection r_α for $\alpha \in \Delta$.

11. Prove that if Φ is irreducible then so is $\Phi^\vee$.

12. Prove that if $0 \neq \alpha \in E$ and the reflection r_α belongs to W, then $r_\alpha = r_\beta$ for some $\beta \in \Phi$.