Instructions: *Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.*

Due on **Friday, May 5**.

Let L denote a semisimple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero, with a chosen Cartan subalgebra H. Write $\Phi \subset H^*$ be the corresponding root system and let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a simple system for Φ. Let W denote the Weyl group of Φ and let Λ^+ denote the associated set of dominant integral weights in H^*.

1. Show that if V is an arbitrary L-module then the sum of its weight spaces is direct. In other words, show that the intersection of any two weight spaces is trivial.
2. Show that if V is an irreducible L-module with at least one nonzero weight space, then V is equal to the sum of its weight spaces. (This sum is direct by the previous exercise.)
3. Construct an irreducible module of $L = \mathfrak{sl}_2(\mathbb{F})$ with no nonzero weight spaces. (That is, carry out the details in Exercise 20.2(c) in the textbook.)
4. Describe weights and maximal vectors for the natural representation of $L = \mathfrak{sl}_{n+1}(\mathbb{F})$ on $\mathbb{F}^{n+1}$, taking H to be the subalgebra of diagonal matrices in L.
5. Describe weights and maximal vectors for the natural representation of $L = \mathfrak{sp}_{2n}(\mathbb{F})$ on $\mathbb{F}^{2n}$, taking H to be the subalgebra of diagonal matrices in L. (Construct $\mathfrak{sp}_{2n}(\mathbb{F})$ as in §1.2 of the textbook.)
6. For each positive integer d, prove that the number of distinct irreducible L-modules $V(\lambda)$ of dimension at most d is finite. Here $V(\lambda)$ denotes the irreducible standard cyclic module of weight $\lambda \in H^*$ described in §20.3 of the textbook.
7. Let $\lambda \in \Lambda^+$. Prove that 0 occurs as a weight of $V(\lambda)$ if and only if λ is a sum of roots.
8. Show that if $\lambda \in \Lambda^+$ then $V(\lambda)^* \cong V(-w_0\lambda)$ where $w_0 \in W$ is the unique element sending all positive roots to negative roots.
9. Use character theory (namely, the results in §22.5 of the textbook) to show that if $L = \mathfrak{sl}_2(\mathbb{F})$ then $V(m) \otimes V(n) \cong V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(m-n)$ for any nonnegative integers $n \leq m$. Here $V(m)$ denotes the irreducible $\mathfrak{sl}_2(\mathbb{F})$-module of dimension $m+1$.
10. Give a direct proof of Weyl's character formula when $L = \mathfrak{sl}_2(\mathbb{F})$.