MATH 5143 - Lecture #1

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We will cover most of Humphreys's textbook:

* Intro. to Lie algebras and Repn. theory

Some other books listed on website.

Grades: all based on ~weekly HW assignments
(no exams)

Lectures: will post slides on course webpage

Lie algebras: today, basic definitions

some important examples

frame guiding classification problems

Very brief motivation: the most interesting groups in physics, geometry, etc. are

Lie groups = (groups that are also manifolds in a compatible way)

↳ e.g. $GL_n(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{Q}_p$

$SL_n(\mathbb{F})$, $O_n(\mathbb{F})$, $Sp_n(\mathbb{F})$ etc.

the most important features in geometry / repn. theory of Lie groups are controlled by the tangent space at the identity element. this tangent space has more structure than just being a vector space — namely it is what we call a Lie algebra.

What is a Lie algebra?

See wikipedia: Lie bracket of vector fields

Constructive definition of Lie algebras:

Let $\mathbb{F}$ be some field (eg. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, etc.)
$\mathbb{F}_p$

Let n be a positive integer and define

$$\mathfrak{gl}_n(\mathbb{F}) = \{ n \times n \text{ matrices over } \mathbb{F} \}$$

For $X, Y \in \mathfrak{gl}_n(\mathbb{F})$ let $[X, Y] = XY - YX$.

Def A (finite-dimensional) Lie algebra is a subspace $L \subseteq \mathfrak{gl}_n(\mathbb{F})$ such that $[X, Y] \in L \;\; \forall X, Y \in L$.

Some examples:

① $L = \mathfrak{gl}_n(\mathbb{F})$ ← call this the general linear Lie alg.

② $L = \{\text{diagonal matrices in } \mathfrak{gl}_n(\mathbb{F})\}$ ← call this $\mathfrak{d}_n(\mathbb{F})$

③ $L = \{\text{upper triangular matrices in } \mathfrak{gl}_n(\mathbb{F})\} = \mathfrak{t}_n(\mathbb{F})$

④ $L = \{\text{strictly upper-}\Delta \text{ matrices in } \mathfrak{gl}_n(\mathbb{F})\} = \mathfrak{n}_n(\mathbb{F})$ "nilpotent"

In fact, any subalgebra $L \subseteq \mathfrak{gl}_n(\mathbb{F})$ (a subspace of matrices closed under multiplication) is a Lie algebra since if $X, Y \in L$ then $[X, Y] = \underbrace{XY}_{\in L} - \underbrace{YX}_{\in L} \in L$

But there are more examples:

Ⓐ $\mathfrak{sl}_n(\mathbb{F}) \overset{\text{def}}{=} \left\{ X \in \mathfrak{gl}_n(\mathbb{F}) \;\middle|\; \operatorname{trace}(X) = \sum_i X_{ii} = 0 \right\}$

"special linear Lie algebra" (not an algebra)

recall that $\operatorname{trace}(XY) = \operatorname{trace}(YX)$

so $\operatorname{trace}([X,Y]) = \operatorname{trace}(XY) - \operatorname{trace}(YX) = 0$.

Ⓒ suppose $n = 2m$ is even

$$\mathfrak{sp}_n(\mathbb{F}) \overset{\text{def}}{=} \left\{ \begin{pmatrix} M & N \\ P & -M^T \end{pmatrix} \;\middle|\; \begin{array}{c} M, N, P \in \mathfrak{gl}_m(\mathbb{F}) \\ N = N^T,\ P = P^T \end{array} \right\}$$

"symplectic Lie algebra"

Ⓓ Suppose $n = 2m$ is even

$$\mathfrak{D}_n(\mathbb{F}) \overset{\text{def}}{=} \left\{ \begin{pmatrix} M & N \\ P & -M^T \end{pmatrix} \;\middle|\; \begin{matrix} M, N, P \in \mathfrak{gl}_m(\mathbb{F}) \\ N^T = -N,\ P^T = -P \end{matrix} \right\}$$

"even orthogonal Lie algebra"

Ⓑ Suppose $n = 2m+1$ is odd

$$\mathfrak{B}_n(\mathbb{F}) \overset{\text{def}}{=} \left\{ \begin{pmatrix} 0 & A & B \\ -A^T & M & N \\ -B^T & P & -M^T \end{pmatrix} \;\middle|\; \begin{matrix} A, B \text{ are } 1 \times m \text{ vectors} \\ M, N, P \in \mathfrak{gl}_m(\mathbb{F}) \\ N^T = -N,\ P^T = -P \end{matrix} \right\}$$

"odd orthogonal Lie algebra"

Examples Ⓐ Ⓑ Ⓒ Ⓓ make up the <u>classical Lie algebras</u>

Later today we will give "basis independent" definitions

We call $[\cdot,\cdot]$ the Lie bracket. Its key properties:

$a_i, b_i \in \mathbb{F}$

$X_i, Y_i \in \mathfrak{gl}_n(\mathbb{F})$

① The Lie bracket is bilinear

$$[a_1X_1 + a_2X_2,\ b_1Y_1 + b_2Y_2] = \sum_{i,j=1}^{2} a_ib_j [X_i, Y_j]$$

$$[a_1X_1 + a_2X_2, Y] = (a_1X_1 + a_2X_2)Y - Y(a_1X_1 + a_2X_2)$$

$$= a_1(X_1Y - YX_1) + a_2(X_2Y - YX_2)$$

② The Lie bracket is alternating: $\underbrace{[X,X]}_{XX - XX = 0} = 0 \ \forall X$

$[X+Y, X+Y] = 0 = [X,X] + [X,Y] + [Y,X] + [Y,Y]$

$0 = [X,Y] + [Y,X]$

③ [Props ①+② $\Rightarrow$] bracket is skew-symmetric: $\underbrace{[X,Y]}_{XY - YX} = -\underbrace{[Y,X]}_{= -(YX - XY)}$

④ Let $\underset{\substack{\parallel \\ [X,\,\cdot\,]}}{ad_X}$ denote the map $\mathfrak{gl}_n(\mathbb{F}) \to \mathfrak{gl}_n(\mathbb{F})$

$$ad_X(Y) = [X, Y]$$

Then it holds that

$$\boxed{ad_{[X,Y]} = [\overbrace{ad_X, ad_Y}^{\substack{= ad_X \circ ad_Y - ad_Y \circ ad_X \\ \text{which is a map } \mathfrak{gl}_n(\mathbb{F}) \to \mathfrak{gl}_n(\mathbb{F})}}] \text{ for all } X, Y}$$

not obvious, (need to prove this)

what does this mean?

short answer: $[f, g] \overset{\text{def}}{=} fg - gf$ whenever f and g are things we can compose/multiply

Long answer: for any vector space V, let $gl(V)$ be space of linear maps $V \to V$

for any $f, g \in gl(V)$ define $[f,g] = f \circ g - g \circ f$
$= fg - gf$

Pf (that $ad_{[X,Y]} = [ad_X, ad_Y]$)

$$ad_{[X,Y]}(Z) = [[X,Y],Z] = [XY - YX, Z]$$
$$= XYZ - YXZ - ZXY + ZYX$$

equal after some cancellation

$$[ad_X, ad_Y](Z) = ad_X(ad_Y(Z)) - ad_Y(ad_X(Z))$$
$$= X(YZ - ZY) - (YZ - ZY)X$$
$$- Y(XZ - ZX) + (XZ - ZX)Y$$

Thus $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$ as linear maps $\mathfrak{gl}_n(\mathbb{F}) \to \mathfrak{gl}_n(\mathbb{F})$

in words: "ad commutes with the Lie bracket" or

"ad is a $\underbrace{\text{Lie algebra homomorphism}}_{\text{need to define what this means}}$ $\mathfrak{gl}_n(\mathbb{F}) \to \underbrace{\mathfrak{gl}(\mathfrak{gl}_n(\mathbb{F}))}_{\text{need to explain how this is a Lie algebra}}$"

Next: defining Lie algebras abstractly.

Suppose L is an $\mathbb{F}$-vector space with a map $[\cdot,\cdot] : L \times L \to L$ (to be called the Lie bracket).

Def L is a Lie algebra with respect to $[\cdot,\cdot]$ if the following conditions hold:

(L1) the bracket is bilinear

(L2) the bracket is alternating: $[x,x]=0 \ \forall x \in L$

(L3) $\mathrm{ad}_{[x,y]} = \mathrm{ad}_x \mathrm{ad}_y - \mathrm{ad}_y \mathrm{ad}_x \overset{\text{def}}{=} [\mathrm{ad}_x, \mathrm{ad}_y]$

for all $x, y \in L$ (here $\mathrm{ad}\, x : L \to L$, $A \mapsto [x, A]$)

Remarks ① L may be infinite dimensional but we will rarely consider this case, the theory is much more involved. Unless stated explicitly, all Lie algebras L are assumed to have $\dim L < \infty$.

② Axioms (L1) + (L2) imply that $[X, Y] = -[Y, X] \ \forall X, Y$

the Lie bracket is always skew symmetric.

it might seem more natural to replace (L2) by skew-symmetry, but this leads to problems when

$\operatorname{char}(\mathbb{F}) = 2$

↖ minimum n such that $\underbrace{1+1+1+\cdots+1}_{n \text{ times}} = 0 \in \mathbb{F}$

or zero if no n exists

③ Axiom L3 $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$

is called the <u>Jacobi identity</u> and is equivalent to

L3' $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

$\forall X, Y, Z \in L$. To remember this:

XYZ
YZX
ZXY
(cyclic permutation)

There are many equivalent ways of stating the Jacobi identity

Usually a given vectorspace has only one natural Lie algebra structure and so we reuse the symbol $[\cdot,\cdot]$ to denote the Lie bracket for any Lie algebra.

Two key concepts: a Lie subalgebra of a Lie algebra L is a subspace $K \leq L$ with $[x,y] \in K$ $\forall x,y \in K$

a linear map $\phi : L_1 \to L_2$ between Lie algebras is a (Lie algebra) morphism if $\phi(\underbrace{[x,y]}_{\text{Lie bracket in } L_1}) = \overbrace{[\phi(x),\phi(y)]}^{\text{Lie bracket in } L_2}$ $\forall x,y$

Fundamental example Let V be an $\mathbb{F}$-vector space.

Then $\mathfrak{gl}(V)$ is a Lie algebra for the bracket $[f,g]=fg-gf$

The Jacobi identity says that $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is a morphism.

A (Lie algebra) isomorphism is a morphism that is a bijection. If $\dim V = n < \infty$ then choosing a basis for V defines an isomorphism $\mathfrak{gl}(V) \xrightarrow{\sim} \mathfrak{gl}_n(\mathbb{F})$

Abstract examples

(A) the trace of $X \in \mathfrak{gl}(V)$ is well-defined whenever $\dim V = n < \infty$, independent of choice of basis.

so we can define $\mathfrak{sl}(V) = \{X \in \mathfrak{gl}(V) \mid \operatorname{trace}(X) = 0\}$

this is a Lie subalgebra of $\mathfrak{gl}(V)$ by same argument as earlier.

BCD Suppose $B : V \times V \to V$ is some bilinear form

[example: if $V = \mathbb{F}^n$ then every B has formula

$B(x,y) = x^T M y$ for some fixed $n \times n$ matrix M]

Then the subspace

$$L \stackrel{\text{def}}{=} \left\{ X \in \mathfrak{gl}(V) \;\middle|\; \begin{array}{c} B(Xu,v) = -B(u,Xv) \\ \forall u,v \in V \end{array} \right\}$$

is a Lie subalgebra of $\mathfrak{gl}(V)$.

Pf. If $X, Y \in L$ then $B([X,Y]u,v) = B(XYu,v) - B(YXu,v)$

$= -B(Yu,Xv) + B(Xu,Yv) = B(u,YXv) - B(u,XYv) =$

$= B(u,[X,Y]v)$ $\forall u,v \in V$ so $[X,Y] \in L$. $\square$

Assume $\dim V = n < \infty$

(+ some more conditions on signature of B)

(B) If B is $\underbrace{\text{symmetric}}_{B(u,v)=B(v,u)}$ and $\underbrace{\text{nondegenerate}}$ then $L \cong O_n(\mathbb{F})$

(D)

[if $u \in V$ is nonzero then $B(u,v) \neq 0$ for some $v \in V$] $\Leftarrow$ [the map $u \mapsto B(u,\cdot)$ is isomorphism $V \cong V^*$]

equivalence is only valid if $\dim V < \infty$

The explicit construction of $\square_n(\mathbb{F})$ earlier corresponds to taking $B(u,v) = u^T M v$ for the matrices

$$M = \left[\begin{array}{c|c} 0 & \begin{matrix} & & 1 \\ & ⋰ & \\ 1 & & \end{matrix} \\ \hline \begin{matrix} & & 1 \\ & ⋰ & \\ 1 & & \end{matrix} & 0 \end{array}\right] = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \quad \text{if } n = 2m \text{ is even}$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_m & 0 \end{bmatrix} \quad \text{if } n = 2m+1 \text{ is odd}$$

(c) If B is skew-symmetric and nondegenerate

$$B(u,v) = -B(v,u)$$

(exercise)

then $\boxed{n = 2m \text{ must be even}}$ and $L \cong \mathfrak{sp}_n(\mathbb{F})$

Explicit construction earlier had $B(u,v) = u^T M v$

for $M = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} = \left[\begin{array}{ccc|ccc} & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \\ \hline -1 & & & & & \\ & \ddots & & & 0 & \\ & & -1 & & & \end{array}\right]$