MATH 5143 - Lecture #2

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From last time: if unspecified, $[X,Y] \overset{\text{def}}{=} XY - YX$

Suppose $\mathbb{F}$ is a field and L is an $\mathbb{F}$-vector space with a map $[\cdot,\cdot] : L \times L \to L$. For $X \in L$ let $\operatorname{ad} X \overset{\text{def}}{=} [X, \cdot]$ denote the map $L \to L$ sending $Y \mapsto [X,Y]$

Def L is a Lie algebra with bracket $[\cdot,\cdot]$ if

1. $[\cdot,\cdot]$ is bilinear
2. $[\cdot,\cdot]$ is alternating: $[X,X] = 0 \ \forall X \in L$
3. $[\operatorname{ad} X, \operatorname{ad} Y] = \operatorname{ad}[X,Y] \ \forall X, Y \in L$ (Jacobi identity)

implies $[X,Y] = -[Y,X]$

A new abstract example: derivations

Suppose A is an $\mathbb{F}$-algebra, possibly non-associative.
(A is just a vector space with a bilinear multiplication)

We have seen that $\underbrace{\mathfrak{gl}(A)}^{\text{space of linear maps } A \to A}$ is a Lie algebra with bracket $[X,Y] = XY - YX$

Let $\operatorname{Der} A = \left\{ \begin{array}{l} \text{linear maps } \delta : A \to A \\ \text{with } \delta(ab) = a\delta(b) + \delta(a)b \ \forall a,b \end{array} \right\}$

Call elems $\delta \in \operatorname{Der} A$ derivations.

Exercise $\operatorname{Der} A$ is a Lie subalgebra of $\mathfrak{gl}(A)$

Should also mention: if A is an associative algebra (meaning $(ab)c = a(bc)$ $\forall a,b,c \in A$) then A can be viewed as a Lie algebra for the bracket $[X,Y] = XY - YX$.

(the fact that this satisfies the Jacobi identity does require associativity of the algebra)

Next: a laundry list of analogies with group theory

Notation: $\operatorname{ad} X : Y \mapsto [X, Y]$ for X, Y in a Lie algebra L

$\operatorname{Ad} g : h \mapsto g^{-1} h g$ for g, h in a group G

Lie algebras L	groups G
an *ideal* of L is a subspace $I \subseteq L$ with $(\operatorname{ad} x)(I) \subseteq I \;\; \forall x \in L$	a *normal subgroup* of G is a subgroup H with $(\operatorname{Ad} g)(H) \subseteq H \;\; \forall g \in G$
(every ideal is a Lie subalgebra)	
the *center* of L is the ideal $Z(L) = \{ Y \in L \mid (\operatorname{ad} x)(Y) = 0 \;\; \forall x \in L \}$	the *center* of G is the normal subgroup $Z(G) = \{ h \in G \mid (\operatorname{Ad} g)(h) = h \;\; \forall g \in G \}$

Lie algebras L

quotient Lie algebra:

given an ideal $I \subseteq L$ the quotient vector space

$$L/I = \{x + I \mid x \in L\}$$

is a Lie algebra for the bracket $[x+I, y+I] \overset{\text{def}}{=} [x,y] + I$ for $x, y \in L$

the derived Lie algebra $[L,L]$ is the span of $\{[x,y] \mid x, y \in L\}$

groups G

quotient group:

given a normal subgroup N the set of cosets

$$G/N = \{gN \mid g \in G\}$$

is a group for usual set product.

the derived subgroup $[G,G]$ is the subgroup generated by $\{ghg^{-1}h^{-1} \mid g, h \in G\}$

Lie algebras L	groups G
L is abelian if $L = Z(L)$	G is abelian if $G = Z(G)$
$\iff$ if $[L,L] = 0$	$\iff$ if $[G,G] = \{1\}$,
$\iff$ $[x,y] = 0 \ \forall\, x,y$	$\iff$ $gh = hg \ \forall\, g,h \in G$
L is simple if L is non-abelian with no proper nonzero ideals	G is simple if G has no proper, nontrivial normal subgroups (can be abelian)

Some other terminology:

the <u>normalizer</u> of a Lie subalgebra $K \subseteq L$

is $N_L(K) = \{ x \in L \mid (\operatorname{ad} x)(K) \subseteq K \}$

(this is a Lie subalgebra, the largest one such that $K \subseteq N_L(K)$ is an ideal)

the centralizer of a subspace $K \subseteq L$

is $C_L(K) = \{ x \in L \mid (\operatorname{ad} x)(K) = 0 \}$

(this is another Lie subalgebra)

Ex. Suppose $L = \mathfrak{sl}_2(\mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a,b,c \in \mathbb{F} \right\}$

Assume $\mathrm{char}(\mathbb{F}) \neq 2$.

A basis for L is $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

The Lie brackets are:

$$[X,X] = [Y,Y] = [H,H] = 0$$

$$[X,Y] = -[Y,X] = H$$

$$[H,X] = -[X,H] = 2X$$

$$[H,Y] = -[Y,H] = -2Y$$

So ad H is diagonalizable

Note: $\mathrm{ad}\, H : Z \mapsto [H,Z]$ has eigenvalues $-2,\ 0,\ 2$

eigenvectors $Y \quad H \quad X$

Claim $L = \mathfrak{sl}_2(\mathbb{F})$ is **simple** when $\mathrm{char}(\mathbb{F}) \neq 2$.

Pf Suppose $0 \neq g \overset{\text{def}}{=} aX + bY + cH \quad (a, b, c \in \mathbb{F})$ belongs to an ideal $I \subseteq L$.

$$[X,[X,g]] = [X, bH - 2cX] = \overbrace{-2b}^{\text{need } \mathrm{char}(\mathbb{F}) \neq 2 \text{ so that this is nonzero}} X \in I$$

$$[Y,[Y,g]] = [Y, aH - 2cY] = -2aY \in I$$

If $a \neq 0$ then $\underbrace{Y \in I}$, but then $H \in I$, so $X \in I = L$.

then $[X,Y] = H \in I$

If $b \neq 0$ then $\underbrace{X \in I}$, then $H \in I$, so $Y \in I = L$.

then $[-Y,X] = [X,Y] = H \in I$

If $a = b = 0$ then $H \in I$, so $X, Y \in I = L$. Thus $I = L$. $\square$

Basic facts about quotients

ⓐ If $\phi : L \to K$ is a surjective Lie algebra morphism then the kernel $\ker\phi = \{x \in L \mid \phi(x) = 0\}$ is an ideal of L and $L/\ker\phi \cong K$

via the map $x + \ker\phi \mapsto \phi(x)$ for $x \in L$.

ⓑ If $I, J \leq L$ are ideals and $I \leq J$ then J/I is ideal of L/I and

$$(L/I)/(J/I) \xrightarrow{\cong} L/J \text{ as Lie algebras}$$

$$(x+I) + J/I \longmapsto x + J$$

ⓒ If $I, J \leq L$ are ideals then $(I+J)/J \cong I/(I \cap J)$

$$(i+j) + J \longmapsto i + I \cap J$$

Terminology: a representation of a Lie algebra L is a Lie algebra morphism $\phi : L \to \mathfrak{gl}(V)$ for some (not necessarily finite-dim. vector space V)

Ex The adjoint representation $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is a representation.

Most interesting Lie algebras arise as subalgebras of $\boxed{\mathfrak{gl}(V)}$.

Prop Any simple Lie algebra is isomorphic to a subalgebra of a general linear Lie algebra

Pf More generally, if $Z(L) \overset{\text{def}}{=} \{x \in L \mid [x,y]=0 \ \forall y\}$

then $Z(L) = \ker(\text{ad})$ so

$L/Z(L) = L/\ker(\text{ad}) \cong \text{ad}(L) \subseteq \mathfrak{gl}(L)$

Therefore $L \cong (\text{subalgebra of } \mathfrak{gl}(L))$ whenever $Z(L)=0$

The center is an ideal so is zero if L is simple

(as simple $\Rightarrow$ non-abelian) $\square$

Derived series of a Lie algebra L: $L^{(0)} = L$

$$L^{(n+1)} = [L^{(n)}, L^{(n)}]$$

Def If $I, J \subseteq L$ then $[I, J]$ is the span of $\left\{ [x,y] \mid \begin{matrix} x \in I \\ y \in J \end{matrix} \right\}$

L is solvable if $L^{(n)} = 0$ for some $n \gg 0$.

Ex If $t_n(\mathbb{F})$ = upper-Δ matrices
$\quad\quad \pi_n(\mathbb{F})$ = strictly upper-Δ matrices
both solvable

as one can check that $t_n(\mathbb{F})^{(1)} = \pi_n(\mathbb{F})$

$$t_n(\mathbb{F})^{(k)} \subseteq \operatorname{span}\{E_{ij} \mid j-i \geq 2^{k-1}\}$$

so $t_n(\mathbb{F})^{(k)} = 0$ if $2^{k-1} > n-1$ so $t_n(\mathbb{F})$ is solvable.

<u>Lower/descending central series</u>: $L^0 = L$

$$L^{n+1} = [L, L^n]$$

L is <u>nilpotent</u> if $L^n = 0$ for some $n \gg 0$.

nilpotent $\Rightarrow$ solvable but not reverse

nilpotent $\downarrow$ strictly upper-Δ

solvable $\downarrow$ upper-Δ

Can show that $t_n(\mathbb{F})$ has $t_n(\mathbb{F})^k = \pi_n(\mathbb{F})$ for all $k \geq 1$.
so $t_n(\mathbb{F})$ is solvable but not nilpotent. But $\pi_n(\mathbb{F})$ is nilpotent.

Next time: a little more discussion of solvable and nilpotent Lie algebras $\rightsquigarrow$ Engel's theorem

then we will discuss the problem of classifying semisimple Lie algebras