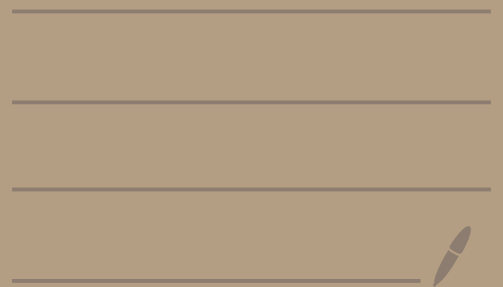


MATH 5143 - Lecture #2



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From last time: if unspecified, $[x, y] \stackrel{\text{def}}{=} xy - yx$

Suppose F is a field and L is an F -vector space with a map $[\cdot, \cdot] : L \times L \rightarrow L$. For $x \in L$ let $\text{ad } x \stackrel{\text{def}}{=} [x, \cdot]$ denote the map $L \rightarrow L$ sending $y \mapsto [x, y]$

Def L is a Lie algebra with bracket $[\cdot, \cdot]$

if ① $[\cdot, \cdot]$ is bilinear \rightarrow implies $[x, y] = -[y, x]$

② $[\cdot, \cdot]$ is alternating: $[x, x] = 0 \forall x \in L$

③ $[\text{ad } x, \text{ad } y] = \text{ad } [x, y] \forall x, y \in L$ (Jacobi identity)

A new abstract example: derivations

Suppose A is an \mathbb{F} -algebra, possibly non-associative.

(A is just a vector space with a bilinear multiplication)

space of linear maps $A \rightarrow A$

We have seen that $\mathfrak{gl}(A)$ is a Lie algebra with

$$\text{bracket } [X, Y] = XY - YX$$

$$\text{Let } \text{Der } A = \left. \begin{array}{l} \text{linear maps } \delta: A \rightarrow A \\ \text{with } \delta(ab) = a\delta(b) + \delta(a)b \quad \forall a, b \end{array} \right\}$$

(all elems $\delta \in \text{Der } A$ **derivations**.)

Exercise $\text{Der } A$ is a Lie subalgebra of $\mathfrak{gl}(A)$

Should also mention: if A is an associative algebra (meaning $(ab)c = a(bc) \forall a, b, c \in A$) then A can be viewed as a Lie algebra for the bracket $[x, y] = xy - yx$.

(the fact that this satisfies the Jacobi identity does require associativity of the algebra)

Next: a laundry list of analogies with group theory

Notation: $\text{ad } x : Y \mapsto [x, Y]$ for x, Y in a Lie algebra L

$\text{Ad } g : h \mapsto g^{-1}hg$ for g, h in a group G

Lie algebras L

an ideal of L is

a subspace $I \subseteq L$ with

$$(\text{ad } x)(I) \subseteq I \quad \forall x \in L$$

(every ideal is a Lie subalgebra)

the center of L is the

ideal $Z(L) = \left\{ Y \in L \mid (\text{ad } x)(Y) = 0 \right.$
 $\left. \forall x \in L \right\}$

groups G

a normal subgroup of G

is a subgroup H with

$$(\text{Ad } g)(H) \subseteq H \quad \forall g \in G$$

the center of G is the

normal subgroup $Z(G) =$

$$\{ h \in G \mid (\text{Ad } g)(h) = h \quad \forall g \in G \}$$

Lie algebras L

quotient Lie algebra:

given an ideal $\mathfrak{I} \subseteq L$

the quotient vector space

$$L/\mathfrak{I} = \{x + \mathfrak{I} \mid x \in L\}$$

is a Lie algebra for the

bracket $[x + \mathfrak{I}, y + \mathfrak{I}] \stackrel{\text{def}}{=} [x, y] + \mathfrak{I}$

$$[x, y] + \mathfrak{I}$$

for $x, y \in L$

the derived Lie algebra $[L, L]$

is the span of $\{[x, y] \mid x, y \in L\}$

groups G

quotient group:

given a normal subgroup N

the set of cosets

$$G/N = \{gN \mid g \in G\}$$

is a group for usual

set product.

the derived subgroup $[G, G]$

is the subgroup generated

$$\text{by } \{ghg^{-1}h^{-1} \mid g, h \in G\}$$

Lie algebras L

L is abelian if $L = Z(L)$

\Leftrightarrow if $[L, L] = 0$

$\Leftrightarrow [x, y] = 0 \forall x, y$

L is simple if L
is non-abelian with
no nonzero ideals
proper

groups G

G is abelian if $G = Z(G)$

\Leftrightarrow if $[G, G] = \{1\}$.

$\Leftrightarrow gh = hg \forall g, h \in G$

G is simple if G has
no proper, nontrivial
normal subgroups

\uparrow
(can be abelian)

Some other terminology:

the normalizer of a Lie subalgebra $K \subseteq L$

$$\text{is } N_L(K) = \{ x \in L \mid (\text{ad } x)(K) \subseteq K \}$$

(this is a Lie subalgebra, the largest one such that $K \subseteq N_L(K)$ is an ideal)

the centralizer of a subspace $K \subseteq L$

$$\text{is } C_L(K) = \{ x \in L \mid (\text{ad } x)(K) = 0 \}$$

(this is another Lie subalgebra)

Ex. Suppose $L = \mathfrak{sl}_2(\mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{F} \right\}$

Assume $\text{char}(\mathbb{F}) \neq 2$.

A basis for L is $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The Lie brackets are:

$$[X, X] = [Y, Y] = [H, H] = 0$$

$$[X, Y] = -[Y, X] = H$$

$$[H, X] = -[X, H] = 2X$$

$$[H, Y] = -[Y, H] = -2Y$$

So $\text{ad } H$ is diagonalizable

Note: $\text{ad } H : Z \mapsto [H, Z]$ has eigenvalues $-2, 0, 2$
eigenvectors Y, H, X

Claim $L = \mathfrak{sl}_2(\mathbb{F})$ is simple when $\text{char}(\mathbb{F}) \neq 2$.

PF Suppose $0 \neq g \stackrel{\text{def}}{=} aX + bY + cH$ ($a, b, c \in \mathbb{F}$)

belongs to an ideal $I \subseteq L$.

need $\text{char}(\mathbb{F}) \neq 2$ so that
this is nonzero

$$[X, [X, g]] = [X, bH - 2cX] = -2bx \in I$$

$$[Y, [Y, g]] = [Y, aH - 2cY] = -2ay \in I$$

if $a \neq 0$ then $Y \in I$, but then $H \in I$, so $X \in I = L$.

if $b \neq 0$ then $X \in I$, then $H \in I$, so $Y \in I = L$.

if $a=b=0$ then $H \in I$, so $X, Y \in I = L$. Thus $I = L$. \square

Basic facts about quotients

(a) If $\phi : L \rightarrow K$ is a surjective Lie algebra morphism then the kernel $\ker \phi = \{x \in L \mid \phi(x) = 0\}$ is an ideal of L and $L / \ker \phi \cong K$

via the map $x + \ker \phi \mapsto \phi(x)$ for $x \in L$.

(b) If $I, J \subseteq L$ are ideals and $I \subseteq J$ then J/I is ideal of L/I and

$(L/I) / (J/I) \cong L/J$ as Lie algebras

$(x+I) + J/I \mapsto x+J$

(c) If $I, J \subseteq L$ are ideals then $(I+J)/J \cong I/(I \cap J)$

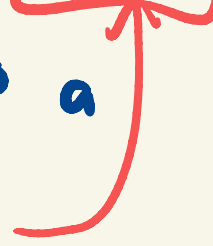
$(i+j) + J \mapsto i + I \cap J$

Terminology: a representation of a Lie algebra L
is a Lie algebra morphism $\phi : L \rightarrow \mathfrak{gl}(V)$
for some (not necessarily finite-dim. vector space V)

Ex The adjoint representation $\text{ad} : L \rightarrow \mathfrak{gl}(L)$
is a representation.

Most interesting Lie algebras arise as subalgebras of $\mathfrak{gl}(V)$

Prop Any simple Lie algebra is isomorphic to a
subalgebra of a general linear Lie algebra



PF More generally, if $Z(L) \stackrel{\text{def}}{=} \{x \in L \mid [x, y] = 0 \forall y\}$

then $Z(L) = \ker(\text{ad})$ so

$$L / Z(L) = L / \ker(\text{ad}) \cong \text{ad}(L) \subseteq \text{gl}(L)$$

Therefore $L \cong$ (subalgebra of $\text{gl}(L)$) whenever $Z(L) = 0$

The center is an ideal so is zero if L is simple

(as simple \Rightarrow non-abelian) \square

Derived series of a Lie algebra L : $L^{(0)} = L$
 $L^{(n+1)} = [L^{(n)}, L^{(n)}]$

Def If $I, J \subseteq L$ then $[I, J]$ is the span of $\{[x, y] \mid \begin{matrix} x \in I \\ y \in J \end{matrix}\}$

L is solvable if $L^{(n)} = 0$ for some $n \gg 0$.

Ex If $\mathfrak{t}_n(\mathbb{F}) =$ upper- Δ matrices
 $\mathfrak{n}_n(\mathbb{F}) =$ strictly upper- Δ matrices } both solvable

as one can check that $\mathfrak{t}_n(\mathbb{F})^{(1)} = \mathfrak{n}_n(\mathbb{F})$
 $\mathfrak{t}_n(\mathbb{F})^{(k)} \subseteq \text{span}\{E_{ij} \mid j-i \geq 2^{k-1}\}$

so $\mathfrak{t}_n(\mathbb{F})^{(k)} = 0$ if $2^{k-1} > n-1$ so $\mathfrak{t}_n(\mathbb{F})$ is solvable.

Lower/descending central series: $L^0 = L$

$$L^{n+1} = [L, L^n]$$

L is nilpotent if $L^n = 0$ for some $n \gg 0$.

nilpotent \Rightarrow solvable but not reverse



strictly
upper- Δ



upper- Δ

Can show that $t_n(\mathbb{F})$ has $t_n(\mathbb{F})^k = \pi_n(\mathbb{F})$ for all $k \geq 1$.
So $t_n(\mathbb{F})$ is solvable but not nilpotent. But $\pi_n(\mathbb{F})$ is nilpotent.

Next time: a little more discussion of solvable and nilpotent Lie algebras \rightsquigarrow Engel's theorem

then we will discuss the problem of classifying semisimple Lie algebras