Math 5143 - Lecture #3

Review from last time

functions, matrices, etc.

Notation: whenever f and g are things we can compose or multiply, we define

$$[f,g] \overset{\text{def}}{=} fg - gf$$

Choose $\mathbb{F}$ to be an arbitrary field (or just set $\mathbb{F} = \mathbb{C}$)

Def. A Lie algebra is an $\mathbb{F}$-vector space L with an $\underbrace{\text{alternating}}_{[X,X]=0\ \forall X\in L \Rightarrow [X,Y] = -[Y,X]}$ bilinear form $[\cdot,\cdot] : L \times L \to L$ satisfying the Jacobi identity

$$\underbrace{\mathrm{ad}}_{\mathrm{ad}\,Z \,=\, [Z,\cdot] : L \to L}[X,Y] = \underbrace{[\mathrm{ad}X, \mathrm{ad}Y]}_{\mathrm{ad}X\,\mathrm{ad}Y - \mathrm{ad}Y\,\mathrm{ad}X} \quad \forall X,Y \in L$$

This definition emphasizes the importance of the adjoint representation of L, which is the map

$$\mathrm{ad} : L \to \mathfrak{gl}(L)$$

where for any vector space V, we write $\mathfrak{gl}(V)$ for the set of all linear maps $V \to V$.

<u>Remark</u> The Jacobi identity $\mathrm{ad}[X,Y] = [\mathrm{ad}\,X, \mathrm{ad}\,Y]$ is equivalent to $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ $\forall\, X, Y, Z \in L$.

Def A <u>morphism</u> of Lie algebras is a linear map $\phi : L_1 \to L_2$ such that $\phi([X,Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in L_1$.

Morphisms can be $\underbrace{\text{injective}}_{\Leftrightarrow \ker\phi \overset{\text{def}}{=} \{X \in L_1 \mid \phi(X)=0\} \text{ is zero}}$, $\overbrace{\text{surjective}}^{\Leftrightarrow \phi(L_1) = L_2}$, or $\underbrace{\text{bijective}}_{\text{isomorphisms}}$

Favorite example: $ad : L \to \mathfrak{gl}(L)$ is Lie algebra morphism

Here $\mathfrak{gl}(L)$ is a Lie algebra with bracket $[X,Y] = XY - YX$.

If A is any associative algebra then we can view A as a Lie algebra in the same way, with bracket $[X,Y] = XY - YX$

If V is a $\mathbb{F}$-vector space with $\dim V = n < \infty$

the choosing a basis for V defines a Lie algebra

isomorphism $\mathfrak{gl}(V) \longrightarrow \mathfrak{gl}_n(\mathbb{F}) \stackrel{\text{def}}{=} \left\{ \begin{array}{c} n \times n \text{ matrices} \\ \text{over } \mathbb{F} \end{array} \right\}$

A constructive way to think about finite-dim Lie algebras is as <u>subalgebras</u> of $\mathfrak{gl}_n(\mathbb{F})$, i.e. as subspaces closed under the Lie bracket. This loses no information:

<u>Thm</u> (Ado, et al.) Every Lie algebra L with $\dim L < \infty$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F})$ for some n.

Basic terminology Suppose L is a Lie algebra

⓪ If $H, K \subseteq L$ are subspaces, then $[H,K]$ = subspace spanned by $\{[x,y] : x \in H, y \in K\}$

① A **(Lie) subalgebra** is a subspace $K \subseteq L$ with $[K,K] \subseteq K$

② An **ideal** $I \subseteq L$ is a subspace with $[L,I] = [I,L] \subseteq I$

this holds as $[x,y] = -[y,x]$

Any ideal is also a subalgebra.

③ The **center** of L is the ideal $Z(L) = \{x \in L \mid [x,y] = 0 \ \forall y \in L\}$

④ L is **abelian** if $Z(L) = L$ or if $[L,L] = 0$ (equivalent properties)

(means $[x,y] = 0 \ \forall x, y \in L$)

⑤ L is **simple** if L is not abelian and has no nonzero ideals except L itself

⑥ If $I \subseteq L$ is an ideal then $L/I = \{x + I \mid x \in L\}$ (quotient vector space) is a Lie algebra for the bracket $[x+I, y+I] = [x,y] + I$ for $x, y \in L$

7. A **representation** of L is a morphism $\phi : L \to \mathfrak{gl}(V)$ for some vector space V

8. The **normalizer** and **centralizer** of a subspace $K \subseteq L$ are the sets $N_L(K) = \{x \in L \mid (\operatorname{ad} x)(K) \subseteq K\}$ and

$$C_L(K) = \{x \in L \mid (\operatorname{ad} x)(K) = 0\}$$

Normalizer is largest subalgebra of L containg K as an ideal

Centralizer is an ideal of $N_K(L)$: if $Y \in N_L(K)$, $X \in C_L(K)$, $Z \in K$

then $\operatorname{ad}[Y,X](Z) = (\operatorname{ad}Y \underbrace{\operatorname{ad}X)(Z)}_{=0} - \overbrace{(\operatorname{ad}X \underbrace{\operatorname{ad}Y)(Z)}_{=[Y,Z] \in K}}^{=0} = 0$

*** end of review ***

Solvable Lie algebras

Define $\begin{cases} L^{(0)} = L \\ L^{(n+1)} = [L^{(n)}, L^{(n)}] \end{cases}$

Recall, if $I, J \subseteq L$ then $[I, J]$ is the span of $\left\{ [x,y] \;\middle|\; \begin{matrix} x \in I \\ y \in J \end{matrix} \right\}$

L is <u>solvable</u> if $L^{(n)} = 0$ for some $n \gg 0$.

<u>Ex</u> If $t_n(\mathbb{F})$ = upper-Δ matrices

$\quad\quad\quad \pi_n(\mathbb{F})$ = strictly upper-Δ matrices

then one can check that $t_n(\mathbb{F})^{(1)} = \pi_n(\mathbb{F})$

$$t_n(\mathbb{F})^{(k)} \subseteq \operatorname{span}\{ E_{ij} \mid j - i \geq 2^{k-1} \}$$

so $t_n(\mathbb{F})^{(k)} = 0$ if $2^{k-1} > n-1$ so $t_n(\mathbb{F})$ is solvable.

Prop L is a Lie algebra. If L is solvable then so are all subalgebras and homomorphic images of L

Pf If $K \leq L$ then $K^{(n)} \leq L^{(n)}$ and $\phi(L)^{(n)} = \phi(L^{(n)})$ if ϕ is a morphism. $\square$

Prop If $I \leq L$ is a solvable ideal and L/I is solvable then L is solvable.

Pf In this case $L^{(n)} \leq I$ for some $n \gg 0$ and $I^{(m)} = 0$ for some $m \gg 0$ so $L^{(m+n)} = 0$. $\square$

Prop If $I, J \subseteq L$ are both solvable ideals then so is $I + J$.

Pf $(I+J)/J \cong \underbrace{I/I\cap J}_{\text{homomorphic image of } I}$ is solvable, as is J $\square$

Cor. If $\dim L < \infty$ then L has a $\boxed{\text{unique}}$ maximal solvable ideal (which is equal to L iff L is solvable)

Pf If S is a maximal solvable ideal of L and $I \subseteq L$ is any solvable ideal then $S+I$ is solvable and contains S, so must be equal to S. Thus if I is $\left[\begin{matrix}\text{maximal}\\\text{solvable}\end{matrix}\right]$ then $S = S+I = I$ $\square$

*** Assume $\dim(L) < \infty$ ***

We denote the unique maximal solvable ideal of a Lie algebra L by $\mathrm{Rad}(L)$, call it the <u>radical</u>

<u>Def</u> L is $\boxed{\text{semisimple}}$ if $\mathrm{Rad}(L) = 0$

that is, if L has no nonzero solvable ideals.

(later will see that semisimple $\Leftrightarrow$ "direct sum of simple")

<u>Fact</u> $L/\mathrm{Rad}(L)$ is semisimple

<u>Pf</u> preimage of any nonzero ideal in $L/\mathrm{Rad}(L)$ is an ideal $I \leq L$ containing $\mathrm{Rad}(L)$ so is not solvable so by propositions, $I/\mathrm{Rad}(L)$ is not solvable.

<u>Fact</u> If L is simple then L is semisimple

<u>Pf</u> If L is simple then $0 \neq [L,L] = L$ so L is not solvable so $\mathrm{Rad}(L)$ is a proper ideal so must be zero. $\square$

Nilpotent Lie algebras

L is nilpotent if $L^n = 0$ for some $n >> 0$

where $L^0 = L^{(0)} = L$, $L^1 = L^{(1)} = [L, L]$

$L^2 = [L, [L, L]] \supseteq L^{(2)}$

$L^3 = [L, [L, [L, L]]]$, $\ldots$,

$L^{n+1} = [L, L^n]$

nilpotent $\Longrightarrow$ solvable

"strictly upper-Δ matrix" "upper-Δ matrix"

Prop. If L is nilpotent then so are all of its subalgebras and homomorphic images.

Pf If $K \subseteq L$ then $K^n \subseteq L^n$
and if $\phi : L \to K$ is a morphism then $\phi(L)^n = \phi(L^n)$ $\square$

Prop If $L/Z(L)$ is nilpotent then L is nilpotent
center

Pf In this case, $L^n \subseteq Z(L)$ for some $n > 0$
and then $L^{n+1} \subseteq [L, Z(L)] = 0$. $\square$

Prop If L is nilpotent and $L \neq 0$ then $Z(L) \neq 0$

Pf If $L^n \neq 0$ and $L^{n+1} = 0$ then $0 \neq L^n \subseteq Z(L)$. $\square$

Prop L is nilpotent if and only if there is some $n \gg 0$ such that $\operatorname{ad} X_1 \operatorname{ad} X_2 \cdots \operatorname{ad} X_n = 0$ (as a map $L \to L$) for all $X_1, X_2, \ldots, X_n \in L$

→ concatenation fg means $f \circ g$

Pf L^n is spanned by elements of the form $(\operatorname{ad} X_1 \operatorname{ad} X_2 \cdots \operatorname{ad} X_n)(Y)$

$$= [X_1, [X_2, [X_3, \ldots, [X_n, Y] \ldots]]] \text{ for } X_i, Y \in L. \quad \square$$

We say that $X \in L$ is ad-nilpotent if $\operatorname{ad} X$ is a nilpotent linear transformation $L \to L$, i.e. $(\operatorname{ad} X)^n = 0$ for some n

Cor If L is nilpotent then every $X \in L$ is ad-nilpotent

Pf Take $X_1 = X_2 = \cdots = X_n = X$ in prop. above. $\square$

Engel's thm: Assume L is Lie algebra with $\dim L < \infty$. Then L is nilpotent if (and only if) every element $X \in L$ is ad-nilpotent.

In other words, L is nilpotent if and only if the image $\operatorname{ad} L \subseteq \mathfrak{gl}(L)$ is a set of nilpotent transformations

Lemma! If $X \in \mathfrak{gl}(V)$ is nilpotent ($X^n = 0$ for $n \gg 0$) then $\operatorname{ad} X$ is nilpotent (as an element of $\mathfrak{gl}(\mathfrak{gl}(V))$)

Pf Let $\lambda_x(Y) = XY$ and $\rho_x(Y) = YX$.

Then λ_x and ρ_x are commuting nilpotent elems of $\mathfrak{gl}(\mathfrak{gl}(V))$

since $\lambda_x \rho_x(Y) = \rho_x \lambda_x(Y) = XYX$. If $X^n = 0$ then $\rho_x^n = \lambda_x^n = 0$

so $(\operatorname{ad} x)^{2n} = (\lambda_x - \rho_x)^{2n} = \sum_k \binom{2n}{k} \underbrace{\lambda_x^k \rho_x^{2n-k}}_{\text{one must be zero}}$

$= 0$ □

Thm Suppose $L \leq \mathfrak{gl}(V)$ is a Lie subalgebra and $0 \neq \dim V < \infty$. Assume that every $X \in L$ is nilpotent (so $X^n = 0$ for some $n > 0$ depending on X). Then there exists $0 \neq v \in V$ with $Xv = 0$ for all $X \in L$.

Pf. Any nilpotent linear transformation X has an eigenvector with eigenvalue zero (take any nonzero column of $X^n \neq 0$ if $X^{n+1} = 0$)

If $\dim L \leq 1$ then can just take $v \in V$ to be any 0-eigenvector of some $0 \neq X \in L$. Suppose $\dim L > 1$ and let $K \leq L$ be a maximal proper Lie subalgebra.

By induction (with $\mathrm{ad}\,K$ and L/K replacing L and V) there is a vector $x \in L - K$ with $[Y, x] \in K$ for all $Y \in K$. $\Bigg\{$ there is nonzero element $x + K \in L/K$ such that $(\mathrm{ad}\,Y)(x+K) = 0 + K$ $\forall Y \in K$

This means that $K \subsetneq N_L(K)$ because $N_L(K) \ni X \notin K$.

Since $K \leq L$ is a maximal proper subalgebra, we must have $L = N_L(K)$ so $K \leq L$ is actually an ideal.

Since K is an ideal, the direct sum $K \oplus \mathbb{F}Z$ is a Lie subalgebra of L for any $Z \in L - K$.

Therefore we must have $L = K \oplus \mathbb{F}Z$ for any $Z \in L-K$ and $\dim L = \dim K + 1$. By induction on $\dim L$, the subspace $W = \{v \in V \mid Yv = 0 \ \forall\, Y \in K\}$ is nonzero and we have $LW \subseteq W$ since if $X \in L$, $Y \in K$, $w \in W$ then

$$YXw = X\underbrace{Yw}_{=0} - \underbrace{[X,Y]}_{\in K}w = 0.$$

Any $Z \in L-K$ acts as a nilpotent linear map $W \to W$ so has a 0-eigenvector $0 \neq v \in W$ with $Zv = 0$. This vector is then a 0-eigenvector for every element $X \in K \oplus \mathbb{F}Z = L$. $\square$

Proof of Engel's thm: [if $\mathrm{ad}\,X$ is nilpotent $\forall X \in L$ then L is nilpotent, assuming $\dim L < \infty$]

Assume every $X \in L$ is ad-nilpotent. (with $\dim L < \infty$)

Then $\mathrm{ad}\,L \leq \mathfrak{gl}(L)$ satisfies conditions of prev thm.

So exists $0 \neq X \in L$ with $(\mathrm{ad}\,Y)(X) = [Y, X] = 0 \; \forall Y \in L$

which means that $Z(L) \neq 0$. But now

$L/Z(L)$ has smaller dimension with all elements

still ad-nilpotent, so by induction $L/Z(L)$ is nilpotent.

Hence by earlier lemma, L is also nilpotent. $\square$

Cor If $\dim V = n < \infty$ and $L \leq \mathfrak{gl}(V)$ consists of all nilpotent elems then there exists a **flag** of vector spaces

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$$

such that $XV_i \subseteq V_{i-1}$ for all i and all $X \in L$.

Equivalently, there exists a basis of V relative to which the matrices of all elements $X \in L$ are strictly upper-Δ.

Pf Set $V_1 = \mathbb{F}v$ where $0 \neq v \in V$ has $Lv = 0$. Then apply induction to image of L in $\mathfrak{gl}(V/V_1)$. $\square$

<u>Cor</u> If $\dim L < \infty$ and L is nilpotent and $K \subseteq L$ is a nonzero ideal then $Z(L) \cap K \neq 0$.

<u>Pf</u> L acts on K by adjoint representation so theorem above implies that there exists $0 \neq x \in K$ with $(\operatorname{ad} y)(x) = [y, x] = 0 \;\forall y \in L$, i.e. x is an nonzero element of $Z(L) \cap K$. $\square$