MATH 5143 - Lecture #4

Suppose L is a Lie algebra over a field $\mathbb{F}$

L is solvable if $L^{(n)} = 0$ for some $n >> 0$

where $L^{(0)} = L$, $L^{(1)} = [L,L]$, $L^{(2)} = [[L,L],[L,L]]$, ...

L is nilpotent if $L^n = 0$ for some $n >> 0$ where

$$L^0 = L, \quad L^1 = [L,L], \quad L^2 = [L,[L,L]], \quad L^3 = [L,[L,[L,L]]], \ldots$$

If $\dim L < \infty$ then L has a unique maximal solvable ideal called the radical, denoted $\mathrm{Rad}(L)$.

Def L is semisimple if ($\dim L < \infty$ and) $\mathrm{Rad}(L) = 0$

Fact Assume $\dim L < \infty$. Then $L/\mathrm{Rad}(L)$ is semisimple and if L is simple then $\mathrm{Rad}(L) = 0$.

Recall: if $X \in L$ then $\operatorname{ad} X$ is map $L \to L$
$$Y \mapsto [X,Y]$$

A map $f: V \to V$ is nilpotent if $f^n = 0$ for some $n \geq 1$

we say that $X \in L$ is ad-nilpotent if $\operatorname{ad} X$ is nilpotent

Fact If L is nilpotent then every $X \in L$ is ad-nilpotent

Engel's thm If $\dim L < \infty$ and every $X \in L$ is ad-nilpotent then L is nilpotent.

Thm If $\dim V = n < \infty$ and $L \leq \mathfrak{gl}(V)$ consists of all nilpotent elements then there exists a basis of V relative to which the matrices of all elements $X \in L$ are strictly upper-Δ.

Semisimple Lie algebras

From now on: $\mathbb{F}$ is an algebraically closed field of characteristic zero (eg can assume $\mathbb{F} = \mathbb{C}$)

L is always a Lie algebra over $\mathbb{F}$ of finite dimension

Thm Assume $L \subseteq \mathfrak{gl}(V)$ is solvable, where $0 < \dim V < \infty$. Then there exists $0 \neq v \in V$ that is an eigenvector of every $x \in L$. That is, such that $xv = \lambda(x)v$ for all $x \in L$, for some linear map $\lambda : X \to \mathbb{F}$.

Pf sketch Mimick proof of theorem above ("nilpotent case")

① Find an ideal $K \subseteq L$ with $\dim L = \dim K + 1$

② By induction V has a common eigenvector for K

③ Check that L stabilizes this common K-eigenspace W
(hardest part of proof)

④ Show that some $Z \in L - K$ has an eigenvector in W
(noting that $L = K \oplus \mathbb{F} Z$) as then we can conclude
that v is an eigenvector of every $X \in L$.

For step ①: as L is solvable, $[L,L] \subsetneq L$, so the quotient $L/[L,L]$ is nonzero and abelian, so any subspace of this quotient is an ideal. Define K to be the preimage in L of any codim 1 subspace of $L/[L,L]$. Then K is a (solvable) ideal of L with $\dim L = \dim K + 1$.

For step ②: because $\mathbb{F}$ is algebraically closed, the theorem definitely holds if $\dim L \leq 1$.

If we assume $\dim L \geq 2$ then by induction there is a linear map $\lambda : K \to \mathbb{F}$ and $0 \neq v \in V$ with $Xv = \lambda(X)v \ \forall X \in K$.

Define $W = \{ w \in V \mid Xw = \lambda(X)w \ \forall X \in K \} \neq 0$.

For step 3, we need to show that L preserves W.
Equivalently, we want to check that if $X \in L$, $Y \in K$, $w \in W$
then $YXw = \lambda(Y)Xw$. But all we know now
is that $YXw = X\underbrace{Yw}_{=\lambda(Y)w} - \underbrace{[X,Y]}_{\in K \text{ ideal}}w = \lambda(Y)Xw - \lambda([X,Y])w$.

So it suffices to check that $\boxed{\lambda([X,Y]) = 0 \ \forall X \in L, Y \in K}$
(assume this for now, proof later)

For step 4: write $L = K \oplus \mathbb{F}Z$ for some $Z \in L - K$.
Since $\mathbb{F}$ is alg. closed, Z has an eigenvector $0 \neq v_0 \in W$.
But v_0 is then an eigenvector for every $X \in L = K \oplus \mathbb{F}Z$. $\square$

Now: Claim $\lambda([X,Y])=0$ for all $X\in L, Y\in K$

where $\lambda : K \to \mathbb{F}$ is function with $Yw = \lambda(Y)w$ $\forall w \in W$ $\forall Y \in K$.

Pf Let $w\in W$. Let $n>0$ be minimal with $w, Xw, X^2w, \dots, X^n w$ linearly dependent. Define $W_i := \mathbb{F}\text{span}[w, Xw, X^2w, \dots, X^{i-1}w]$ so that $W_0=0$ and $\dim W_i = \min(i,n)$.

Claim that $YX^iw \in \lambda(Y)X^iw + W_i$

Clear if $i=0$ and if $i>0$ then

$$YX^iw = YXX^{i-1}w = X\underbrace{YX^{i-1}w}_{\substack{\in\lambda(Y)X^{i-1}w+W_{i-1}\\ \text{by induction}}} - \underbrace{[X,Y]X^{i-1}w}_{\substack{\in\lambda([X,Y])X^{i-1}w+W_{i-1}\subseteq W_i\\ \text{by induction}}} \in \lambda(Y)X^iw+W_i$$

Conclude that relative to the basis $w, Xw, X^2w, \dots, X^{i-1}w$ the matrix of Y is $\begin{bmatrix} \lambda(Y) & & * \\ & \lambda(Y) & \\ 0 & & \ddots & \\ & & & \lambda(Y) \end{bmatrix}$

Thus $\text{trace}_{W_n}(Y) = n\lambda(Y)$ (for any $Y \in K$)

But then $\text{trace}_{W_n}(\underbrace{[X, Y]}_{\in K}) = n\lambda([X, Y])$

Since X and Y preserve W_n, the products XY and YX also preserve W_n, so $\text{trace}_{W_n}([X, Y])$

$= \text{trace}(XY) - \text{trace}(YX) = 0.$ Since $\text{char}(\mathbb{F}) = 0$, we have $n\lambda([X, Y]) = 0 \Rightarrow \lambda([X, Y]) = 0$ $\square$

Lie's theorem Suppose $L \subseteq \mathfrak{gl}(V)$ is a solvable Lie algebra where $\dim V = n < \infty$. Then there is some basis of V relative to which the matrices of all elements of L are upper-Δ.

Pf Choose $0 \neq v_1 \in V$ with $xv_1 = \lambda(x)v_1$ $\forall x \in L$ for some linear map $\lambda : L \to \mathbb{F}$. Set $V_1 = \mathbb{F}v_1$ and apply theorem by induction to V/V_1. This gives a basis $v_2 + V_1, v_3 + V_1, \ldots, v_n + V_1$ for V/V_1 and the desired basis is then $v_1, v_2, v_3, \ldots, v_n$. $\square$

Cor Suppose L is solvable with $n = \dim L \, (< \infty)$.

Then there exists a chain of ideals

$$0 = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_n = L \text{ with } \dim L_i = i.$$

a homomorphic image of L, hence also solvable

Pf Apply Lie's theorem to Lie algebra $\operatorname{ad} L \subseteq \mathfrak{gl}(L)$ to get a basis $v_1, v_2, \ldots, v_n \in L$ such that $(\operatorname{ad} x) v_i \in \mathbb{F}\text{-span}[v_1, v_2, \ldots, v_i]$ $\forall x \in L$ $\forall i$

which means that $L_i \overset{\text{def}}{=} \mathbb{F}\text{-span}[v_1, v_2, \ldots, v_i]$ is an ideal. $\square$

<u>Cor</u>. Suppose L is solvable. Then $[L,L]$ is nilpotent

<u>Pf</u> Choose a basis of L such that matrices of

$\mathrm{ad}\, X \in \mathfrak{gl}(L)$ for every $X \in L$ are upper-Δ.

The matrix of $\mathrm{ad}[X,Y] = [\mathrm{ad}\,X, \mathrm{ad}\,Y]$ is then strictly upper-Δ $\forall\, X, Y \in L$, so $\mathrm{ad}\, Z$ is nilpotent for all $Z \in [L,L]$. Engel's theorem then implies that $[L,L]$ is nilpotent. $\square$

Remarks (Assuming $\mathbb{F}$ alg. closed of char. zero and $L \subseteq gl(V)$ where $\dim V < \infty$)

① Have shown that if L is solvable, $\exists$ basis of V that makes L upper-Δ

② If L is nilpotent, then L is solvable. But in this case L does not have to be strictly upper-Δ in basis from ①

<u>Ex</u> $L = $ (all diagonal matrices) $\leftarrow$ abelian Lie algebra

③ We only showed that V must have basis that makes L strictly upper-Δ when L consists of all nilpotent elems, but L can be nilpotent as a Lie algebra without this (consider example)

Jordan-Chevalley decomposition

(for just this topic char $\mathbb{F}$ may be nonzero)

Let V be a finite-dim. vector space

We say that $X \in \mathfrak{gl}(V)$ is **semisimple** if X is **diagonalizable**

meaning V has a basis of eigenvectors for X

(when $\mathbb{F}$ is not alg. closed, "semisimple" means that the roots of minimal polyn. of X are distinct)

Exercises: ① If X and Y commute and are both semisimple, then all linear combinations $aX + bY$ ($a, b \in \mathbb{F}$) are semisimple

② If X is semisimple and X preserves $W \subseteq V$, then $X|_W$ is semisimple

Prop Consider some element $X \in \mathfrak{gl}(V)$

(a) There are unique elements $X_s, X_n \in \mathfrak{gl}(V)$ with X_s semisimple, X_n nilpotent, $X_s X_n = X_n X_s$ and $\boxed{X_s + X_n = X}$. Call this the Jordan (Chevalley) decomposition of X

(idea: if $V = \mathbb{F}^n$ and the Jordan canonical form of the matrix of X has blocks $\begin{bmatrix} a & 1 & & 0 \\ & a & 1 & \\ & & a & \ddots \\ 0 & & & \ddots\ 1 \\ & & & a \end{bmatrix}$ then the Jordan canonical forms of X_s and X_n are $\underbrace{\begin{bmatrix} a & & 0 \\ & a & \\ 0 & & \ddots\ a \end{bmatrix}}_{\text{diagonal}}$ and $\underbrace{\begin{bmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}}_{\text{nilpotent}}$ respectively.)

commute

(b) There are polynomials $p(T), q(T)$ in a variable T with no constant term (so $p(T), q(T) \in T\mathbb{F}[T]$) such that $X_s = p(X)$ and $X_n = q(X)$.

$\Rightarrow$ X_s and X_n commute with any $Y \in \mathfrak{gl}(V)$ that has $XY = YX$.

(c) If $A \subseteq B \subseteq V$ are subspaces with $XB \subseteq A$ then $X_s B \subseteq A$ and $X_n B \subseteq A$.

(this part is immediate from (b))

(omit proof of this $\rightsquigarrow$ standard linear algebra, we may sketch argument in HW exercises)

Prop Suppose V has $\dim V < \infty$ and $X \in \mathfrak{gl}(V)$.

(a) If X is nilpotent then so is $\operatorname{ad} X \in \mathfrak{gl}(\mathfrak{gl}(V))$

(shown in lemma earlier today)

(b) If X is semisimple then so is $\operatorname{ad} X$.

Pf If $v_1, v_2, \ldots, v_n$ is basis of V consisting of eigenvectors for X, so $Xv_i = a_i v_i$ for some $a_i \in \mathbb{F}$, and e_{ij} is the corresponding basis of $\mathfrak{gl}(V)$ so that

e_{ij} is the linear map with $v_k \longmapsto \begin{cases} v_i & \text{if } j=k \\ 0 & \text{if } j\neq k \end{cases}$

then $(\operatorname{ad} X)(e_{ij}) = (a_i - a_j)e_{ij}$, since

$$(\operatorname{ad} X)(e_{ij})(v_k) = [X, e_{ij}](v_k)$$

$$= \underbrace{X e_{ij}(v_k)}_{a_i e_{ij}(v_k)} - \underbrace{e_{ij} \underbrace{X(v_k)}_{= a_k v_k}}_{= a_j e_{ij}(v_k)}$$

$$= (a_i - a_j) e_{ij}(v_k) \qquad \square$$

Lemma Let $X \in \mathfrak{gl}(V)$ where $\dim V < \infty$. Suppose the Jordan decomp of X is $X = X_s + X_n$, then the Jordan decomp of $\operatorname{ad} X$ is

$$\boxed{\operatorname{ad} X = \operatorname{ad} X_s + \operatorname{ad} X_n}$$

Pf $[\operatorname{ad} X_s, \operatorname{ad} X_n] = \operatorname{ad}[X_s, X_n] = 0$

and we already saw that $\operatorname{ad} X_s$ is semisimple, $\operatorname{ad} X_n$ is nilpotent $\square$

Next time: Criteria for solvability and semisimplicity