Math 5143 - Lecture 5

Jordan–Chevalley decomposition

(for just this topic char $\mathbb{F}$ may be nonzero)

Let V be a finite-dim. vector space

We say that $X \in \mathfrak{gl}(V)$ is semisimple if X is diagonalizable

meaning V has a basis of eigenvectors for X

(when $\mathbb{F}$ is not alg. closed, "semisimple" means that the roots of minimal polyn. of X are distinct)

Exercises: ① If X and Y commute and are both semisimple, then all linear combinations $aX + bY$ $(a, b \in \mathbb{F})$ are semisimple

② If X is semisimple and X preserves $W \subseteq V$, then $X|_W$ is semisimple

Prop Consider some element $X \in \mathfrak{gl}(V)$

(a) There are unique elements $X_s, X_n \in \mathfrak{gl}(V)$ with X_s semisimple, X_n nilpotent, $X_s X_n = X_n X_s$ and $\boxed{X_s + X_n = X}$. Call this the Jordan (Chevalley) decomposition of X

(idea: if $V = \mathbb{F}^n$ and the Jordan canonical form of the matrix of X has blocks $\begin{bmatrix} a & 1 & & 0 \\ & a & 1 & \\ & & a & \ddots \\ 0 & & & a \end{bmatrix}$ then the Jordan canonical forms of X_s and X_n are $\underbrace{\begin{bmatrix} a & & 0 \\ & a & \\ 0 & & \ddots \; a \end{bmatrix}}_{\text{diagonal}}$ and $\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ 0 & & \ddots \; 0 \end{bmatrix}}_{\text{nilpotent}}$ respectively.)

commute

(b) There are polynomials $p(T), q(T)$ in a variable T with no constant term (so $p(T), q(T) \in T\mathbb{F}[T]$) such that $X_s = p(X)$ and $X_n = q(X)$.

$\Rightarrow$ X_s and X_n commute with any $Y \in \mathfrak{gl}(V)$ that has $XY = YX$.

(c) If $A \subseteq B \subseteq V$ are subspaces with $XB \subseteq A$ then $X_s B \subseteq A$ and $X_n B \subseteq A$.

(this part is immediate from (b))

(omit proof of this $\rightsquigarrow$ standard linear algebra, we may sketch argument in HW exercises)

Prop Suppose V has $\dim V < \infty$ and $X \in \mathfrak{gl}(V)$.

(a) If X is nilpotent then so is $\operatorname{ad} X \in \mathfrak{gl}(\mathfrak{gl}(V))$

(shown in lemma last week)

(b) If X is semisimple then so is $\operatorname{ad} X$.

Pf If $v_1, v_2, \dots, v_n$ is basis of V consisting of eigenvectors for X, so $Xv_i = a_i v_i$ for some $a_i \in \mathbb{F}$, and e_{ij} is the corresponding basis of $\mathfrak{gl}(V)$ so that

e_{ij} is the linear map with $v_k \longmapsto \begin{cases} v_i & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$

then $(\mathrm{ad}\, X)(e_{ij}) = (a_i - a_j) e_{ij}$, since

$$(\mathrm{ad}\, X)(e_{ij})(v_k) = [X, e_{ij}](v_k)$$

$$= \underbrace{X e_{ij}(v_k)}_{a_i e_{ij}(v_k)} - \underbrace{e_{ij} \underbrace{X(v_k)}_{= a_k v_k}}_{= a_j e_{ij}(v_k)}$$

$$= (a_i - a_j) e_{ij}(v_k) \quad \square$$

__Useful fact__ If $X \in L \subseteq \mathfrak{gl}(V)$ then the Jordan decomposition of $\operatorname{ad} X \in \mathfrak{gl}(L)$ is just

$$\operatorname{ad} X = \underbrace{\operatorname{ad}(X_s)}_{\text{still semisimple}} + \underbrace{\operatorname{ad}(X_n)}_{\text{still nilpotent}}$$

where $X = X_s + X_n$ is the Jordan decomp. of X

__Pf.__ Jacobi identity $\Rightarrow [\operatorname{ad}(X_s), \operatorname{ad}(X_n)] = \operatorname{ad}[X_s, X_n] = 0$ so this holds by uniqueness of Jordan decomposition $\square$

Back to usual hypotheses: $\mathbb{F}$ alg. closed with $\operatorname{char}(\mathbb{F}) = 0$

__Today__: Cartan's criterion for solvability:

Let $L \subseteq \mathfrak{gl}(V)$ where $\dim V < \infty$. If $\operatorname{trace}(XY) = 0$ for all $X \in [L, L]$, $Y \in L$ then L is solvable.

Lemma Let $A \subseteq B$ be two subspaces of $\mathfrak{gl}(V)$ where $\dim V < \infty$. Define

$$M = \{ x \in \mathfrak{gl}(V) \mid [x, B] \subseteq A \}$$

and suppose $x \in M$ has $\operatorname{trace}(xy) = 0 \ \forall y \in M$.

Then x is nilpotent (i.e. $x^n = 0$ for $n \gg 0$)

Pf Write the Jordan decomp of x as $x = x_s + x_n$.

Let $v_1, v_2, v_3, \ldots, v_n$ be a basis for V such that $x_s v_i = a_i v_i$ for some $a_i \in \mathbb{F}$.

Define $E = \mathbb{Q}\text{-span}\{a_1, a_2, \dots, a_n\} \subseteq \mathbb{F}$ rational vector space

We want to show that $x_s = 0$ as then $x = x_n$ is nilpotent.

This holds if and only if $a_1 = a_2 = \dots = a_n = 0 \iff E = 0$.

Let $E^* = \{\mathbb{Q}\text{-linear maps } E \to \mathbb{Q}\}$.

Since $\dim_{\mathbb{Q}} E \leq n < \infty$, it holds that $\dim_{\mathbb{Q}} E^* = \dim_{\mathbb{Q}} E$

So it suffices to show that $E^* = 0$.

Suppose $f \in E^*$. Let $Y \in \mathfrak{gl}(V)$ have matrix $\begin{bmatrix} f(a_1) & & & \\ & f(a_2) & & \\ & & \ddots & \\ & & & f(a_n) \end{bmatrix}$

in basis $v_1\ v_2\ v_3 \dots v_n$ for V.

Then $Yv_i = f(a_i)v_i \ \forall i$. If $e_{ij} \in \mathfrak{gl}(V)$: $v_k \mapsto \begin{cases} v_i & j = k \\ 0 & j \neq k \end{cases}$

then $(\operatorname{ad} x_s)e_{ij} = (a_i - a_j)e_{ij}$ and $(\operatorname{ad} Y)e_{ij} = (f(a_i) - f(a_j))e_{ij}$.

Claim: We can find a polynomial $r(T) \in \mathbb{Q}[T]$ such that $r(a_i - a_j) = f(a_i) - f(a_j) = f(a_i - a_j)$ $\forall i,j$

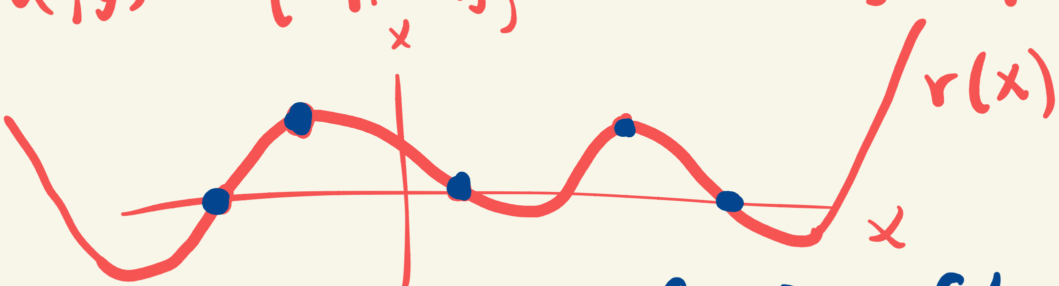
clear by polynomial interpolation

since $f \in E^*$ is linear

$\in \mathbb{Q}$

[The polynomial $r(T)$ is some polynomial that passes through the points

$(x, y) = (a_i - a_j, f(a_i - a_j))$ $\forall i,j$.]



We must have $r(0) = f(0) = f(a_i - a_i) = 0$

so $r(T)$ has no constant term, $r(T) \in T\mathbb{Q}[T]$

Also, we have $\mathrm{ad}\, Y = r(\mathrm{ad}\, X_s)$ since both sides give same result applied to each e_{ij}

Finally by earlier facts, $\mathrm{ad}\, X_s$ is semisimple part of $\mathrm{ad}\, X$ so $\mathrm{ad}\, X_s = p(\mathrm{ad}\, X)$ for some polynomial without constant term.

We assume $(\mathrm{ad}\, X)(B) \subseteq A$ so

$$(\mathrm{ad}\, Y)(B) = r(\mathrm{ad}\, X_s)(B) = r(p(\mathrm{ad}\, X))(B) \subseteq A$$

This implies that $Y \in M = \{ Z \in gl(V) \mid [Z, B] \subseteq A \}$.

Hence by assumption $\mathrm{trace}(XY) = 0$.

But $\mathrm{trace}(\underbrace{XY}) = \sum_{i=1}^{n} a_i \underbrace{f(a_i)}_{\in \mathbb{Q}} \in E = \mathbb{Q}\text{-span}[a_1, a_2, \dots a_n]$

in basis $v_1, v_2, \dots v_n$

$$XY = (X_s + X_n)Y = \underbrace{\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \begin{bmatrix} f(a_1) & & \\ & \ddots & \\ & & f(a_n) \end{bmatrix}}_{\text{trace is } \sum_i a_i f(a_i)} + \underbrace{\begin{bmatrix} 0 & & * \\ & 0 & \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} f(a_1) & & \\ & \ddots & \\ & & f(a_n) \end{bmatrix}}_{\text{trace is zero}}$$

Thus $0 = f(0) = f(\mathrm{trace}(XY)) = \sum_{i=1}^{n} f(a_i)^2$ which can only hold if $f = 0$. Thus $E^* = 0$ $\square$

Prop If $X, Y, Z \in \mathfrak{gl}(V)$, $\dim V < \infty$, then

$$\operatorname{trace}(\underbrace{[X,Y]Z}) = \operatorname{trace}(\underbrace{X[Y,Z]})$$

Pf $= \operatorname{trace}(XYZ) - \underbrace{\operatorname{trace}(YXZ)}_{} \qquad = \operatorname{trace}(XYZ) - \underbrace{\operatorname{trace}(XZY)}_{}$ $\square$

equal

Now can prove:

Cartan's criterion Let $L \subseteq \mathfrak{gl}(V)$ where $\dim V < \infty$.

If $\operatorname{trace}(XY) = 0$ for all $X \in [L, L]$, $Y \in L$ then L is solvable.

Pf To show L is solvable, enough to check that $[L,L]$ is nilpotent, and for this it suffices to check that every $X \in [L,L]$ is nilpotent as an element of $\mathfrak{gl}(V)$.

(this will imply that ad X is nilpotent, so we can use Engel's thm)

Let $A = [L,L] \subseteq B = L$ and $M = \{X \in \mathfrak{gl}(V) \mid [X,L] \subseteq [L,L]\}$

Suppose $X \in [L,L]$ and note that $[L,L] \subseteq M$.

If we can show that $\operatorname{trace}(XY) = 0$ for all $Y \in M$ then lemma implies that is nilpotent. We assume $\operatorname{trace}(XY) = 0$ for all $Y \in L$. Note that $L \subseteq M$. But X is a linear comb. of elems $[X_1, X_2]$ for $X_i \in L$ and if $Y \in M$ then

$$\operatorname{trace}([X_1,X_2]Y) \underset{\uparrow \text{ by prop.}}{=} \operatorname{trace}(X_1[X_2,Y]) \underset{\uparrow \text{ by properties of trace}}{=} \operatorname{trace}(\underbrace{[X_2,Y]}_{\in [L,L] \text{ by def of } M} X_1) \overset{\text{by assumption}}{=} 0 \quad \square$$