# Math 5143 - Lecture 6

From last time: assume $L$ is finite dim Lie algebra / algebraically closed field $\mathbb{F}$ of char. zero.

Cartan's Criterion: $L \subseteq \mathfrak{gl}(V)$ is solvable if $\operatorname{trace}(XY) = 0$ $\forall X \in [L,L], Y \in L$

**Cor.** If $L$ is a Lie algebra such that $\operatorname{trace}(\operatorname{ad} X \operatorname{ad} Y) = 0$ $\forall X \in [L,L]$ $\forall Y \in L$

then $L$ is solvable.

**Pf** Apply Cartan's criterion to $\boxed{\operatorname{ad} L \subseteq \mathfrak{gl}(L)}$

As $\operatorname{ad}[L,L] = [\operatorname{ad} L, \operatorname{ad} L]$, we find that $\operatorname{ad} L$ is solvable. But $\ker \operatorname{ad} = Z(L)$ is solvable so $L$ is solvable since $L/Z(L) \cong \operatorname{ad} L$. $\square$

(by lemma in prev. lecture)

# Killing form

**Def** The bilinear form $\kappa : L \times L \to L$ defined by

$$\kappa(X,Y) \overset{\text{def}}{=} \operatorname{trace}(\operatorname{ad}X \operatorname{ad}Y) \text{ for } X,Y \in L$$

is called the Killing form of $L$.

**Prop** $\kappa$ is $\boxed{\text{symmetric}}$: $\kappa(X,Y) = \kappa(Y,X)$

$\kappa$ is $\boxed{\text{associative}}$: $\kappa([X,Y],Z) = \kappa(X,[Y,Z])$

equivalent to prop. stated last time

[To compute $\kappa(X,Y)$ need to pick a basis of $L$ and write down the matrices of $\operatorname{ad}X$ and $\operatorname{ad}Y$, doesn't matter which basis you choose]

**Lemma** Let $I \subseteq L$ be an ideal.

Then the Killing form $\kappa_I$ of $I$ is the restriction of the Killing form $\kappa = \kappa_L$ of $L$. Thus $\underbrace{\kappa_I(x,y)}_{\substack{\text{trace of} \\ \text{a linear map} \\ I \to I}} = \underbrace{\kappa_L(x,y)}_{\substack{\text{trace of} \\ \text{a linear map} \\ L \to L}}$ $\forall x,y \in I$

**Pf** If $\phi : V \to W \subseteq V$ is a linear map ($\dim V < \infty$)

then $\text{trace}_V(\phi) = \text{trace}_W(\phi|_W)$ because

if $w_1, w_2 \cdots w_k$ is a basis of $W$ and $w_{k+1}, \ldots, w_n$

extends this to a basis of $V$ then the matrix of $\phi$

has form

$$\begin{array}{rc|c} k \text{ rows} & \boxed{\begin{matrix} * & * & * \\ * & * & * \end{matrix}} & \begin{matrix} * & * & * \\ * & * & * \end{matrix} \\ \hline n-k \text{ rows} & 0 & 0 \\ & \underbrace{\phantom{***}}_{k \text{ cols}} & \underbrace{\phantom{***}}_{n-k \text{ cols}} \end{array}$$

is matrix of $\phi|_W$

To prove lemma, apply this observation with $V = L$ and $W = I$. $\square$

The radical of [any symmetric bilinear form] $\kappa : L \times L \to L$

is $S = \{ x \in L \mid \kappa(x,y) = 0 \; \forall y \in L \}$

$= \{ y \in L \mid \kappa(x,y) = 0 \; \forall x \in L \} \supseteq Z(L)$ since

$\operatorname{ad} x = 0 \; \forall x \in Z(L)$

The form $\kappa$ is non-degenerate if $S = 0$.

This happens iff $\boxed{\kappa(x, \cdot) : L \to L \text{ is zero map iff } x = 0}$

or $\boxed{\underbrace{\left[ \kappa(x_i, x_j) \right]}_{n \times n \text{ matrix}}{}_{1 \le i,j \le n} \text{ is invertible for some/any basis } x_1, x_2, \ldots, x_n \in L}$

__Ex__ Suppose $L = \mathfrak{sl}_2(\mathbb{F}) = \mathbb{F}\text{-span}[E, H, F]$

for $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

In this ordered basis we have

$$\left.\begin{aligned} &[E,E] = [H,H] = [F,F] = 0 \\ &[E,H] = -2E \\ &[F,H] = 2F \\ &[E,F] = H \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \operatorname{ad} E &= \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ \operatorname{ad} H &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\ \operatorname{ad} F &= \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \end{aligned}\right.$$

$\boxed{\text{assuming } \operatorname{char}(\mathbb{F}) \neq 2}$

So we have $\left[\kappa(x_i, x_j)\right]_{1 \le i,j \le 3} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{bmatrix}$ which is invertible

$(x_1 = E,\ x_2 = H,\ x_3 = F)$

So killing form of $\mathfrak{sl}_2(\mathbb{F})$ is nondegen.

Suppose $L$ is a Lie algebra.

Recall that $L$ is semisimple if it has no nonzero solvable ideals, i.e., its radical ideal $\mathrm{Rad}(L) = \begin{pmatrix} \text{unique maximal} \\ \text{solvable ideal} \end{pmatrix}$ is zero.

Thm $L$ is semisimple if and only if the Killing form $\kappa$ is nondegenerate.

Thus to check if $L$ is semisimple, just need to pick a basis $x_1, x_2, \dots, x_n$ for $L$ and check if matrix $[\kappa(x_i, x_j)]_{1 \le i,j \le n}$ has nonzero determinant

Pf (that $\operatorname{Rad} L = 0$ iff radical $S$ of $\kappa$ is zero)

The radical $S$ of killing form is an ideal of $L$

(check this using associativity of $\kappa$)

In fact, $\operatorname{ad}_L S$ is a solvable ideal of $\operatorname{ad} L$

by Cartan's criterion:

$$\operatorname{trace}(\operatorname{ad} X \operatorname{ad} Y) = \kappa(X, Y) = 0 \quad \forall X \in S \supseteq [S, S]$$
$$\forall Y \in L \supseteq S$$

The center $Z(S)$ is abelian, hence solvable.

As $\operatorname{ad}_L S \cong S / Z(S)$ we conclude that $S$ is solvable.

Thus if $\operatorname{Rad} L = 0$ then $S = 0$ as $S \subseteq \operatorname{Rad} L$.

Want now to show that $\boxed{S = 0}$ implies that $\boxed{\mathrm{Rad}\,L = 0}$.

It suffices to check that if $I \subseteq L$ is any abelian ideal then $I \subseteq S$. This is because if $I$ is a nonzero solvable ideal, then $I^{(n)}$ is abelian for $n$ such that $I^{(n)} \neq 0 = I^{(n+1)}$

(so if there are no nonzero abelian ideals, there are also no nonzero solvable ideals) Assume $I$ is an abelian ideal.

If $X \in I$, $Y \in L$ then $\mathrm{ad}X\,\mathrm{ad}Y$ is a map $L \xrightarrow{\mathrm{ad}Y} L \xrightarrow{\mathrm{ad}X} I$

so $(\mathrm{ad}X\,\mathrm{ad}Y)^2$ is a map $L \to [I, I] = 0$.

Thus $\mathrm{ad}X\,\mathrm{ad}Y$ is nilpotent so it must have zero trace.

meaning that $\kappa(X, Y) = 0 \Longrightarrow \boxed{I \text{ abelian} \subseteq S}$. $\square$

nilpotent $\Rightarrow$ only eigenvalue is zero $\Rightarrow$ trace = sum of eigvals = 0

# Simple ideals + (semisimple $\equiv$ sum-of-simple)

A Lie algebra $L$ is a direct sum of ideals $I_1, I_2, \dots, I_n$ if $L = I_1 \oplus I_2 \oplus \dots \oplus I_n$.
(as vector spaces)

This would mean that each $x \in L$ has a unique expression as

$$x = x_1 + x_2 + \dots + x_n$$

with $x_j \in I_j$. Uniqueness here implies that $I_j \cap I_k = 0 \ \forall j \neq k$

Since each $I_j$ is an ideal, we must also have $[I_j, I_k] = 0$ $\forall j \neq k$.

**Thm** Suppose $L$ is a semisimple Lie algebra. [As usual, $\dim L < \infty$]

Then there exist ideals $L_1, L_2, \ldots, L_n \leq L$ such that

① each $L_i$ is simple

② $L = L_1 \oplus L_2 \oplus \cdots \oplus L_n$

③ any simple ideal of $L$ is equal to some $L_i$

④ the killing form of $L_i$ is just the restriction of the killing form of $L$ [earlier Lemma already checked this]

**Pf** Let $I$ be any ideal of $L$. Write $\kappa$ for killing form and define $I^\perp = \{x \in L \mid \kappa(x, y) = 0 \ \forall y \in I\}$.

First claim to show is that (a) $I^\perp$ is an ideal & (b) $L = I \oplus I^\perp$.

(a) To see that $I^\perp$ is an ideal, let $X \in L$, $Y \in I^\perp$, $Z \in I$.

Then $\kappa([X,Y],Z) \overset{\text{by skew-symmetry of } [\cdot,\cdot]}{=} -\kappa([Y,X],Z) \overset{\text{by associativity of } \kappa}{=} -\kappa(\underbrace{Y}_{\in I^\perp}, \underbrace{[X,Z]}_{\in I}) = 0$

So we conclude (as $Z$ is arbitrary) that $[X,Y] \in I^\perp$.

(b) because $L$ is semisimple, its center $Z(L)$ is zero,

so $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is injective.

Cartan's criterion applied to $I \cap I^\perp \cong \mathrm{ad}(I \cap I^\perp) \subseteq \mathfrak{gl}(L)$

implies that $I \cap I^\perp$ is solvable:

$\forall X \in [I \cap I^\perp, I \cap I^\perp]$, $\forall Y \in I \cap I^\perp$ we have

$\mathrm{trace}(\mathrm{ad}X\,\mathrm{ad}Y) \overset{\text{def of } \kappa}{=} \kappa(X,Y) \overset{\text{by def of } I^\perp}{=} 0$

So $I \cap I^\perp = 0$

as $\mathrm{Rad}\,L = 0$

Conclude that

$L = I \oplus I^\perp$

(since $\dim L = \dim I + \dim I^\perp$)

So our claims (a) and (b) both hold.

Now we proceed by induction (on $\dim L$).

If $L$ has no nonzero proper ideals then $L$ is simple.

Otherwise we can find a minimal proper nonzero ideal $L_1 \subseteq L$.

Any ideal of $L_1$ is an ideal of $L = L_1 \oplus L_1^\perp$ $\left( [x,y]=0 \ \forall x \in L_1 \ \forall y \in L_1^\perp \right)$
so $L_1$ must be simple itself.

Likewise $L_1^\perp$ must be semisimple since any of its (solvable) ideals are (solvable) ideals of $L$.

Thus by induction we can write $L_1^\perp = L_2 \oplus \dots \oplus L_n$
for simple ideals $L_i$ and then $L = L_1 \oplus L_2 \oplus \dots \oplus L_n$.

This proves 1 + 2, 4 already known.

We still have to prove that if $I$ is any simple ideal of $L$ then $I = L_i$ for some $i \in \{1, 2, \dots, n\}$.

To prove this, we observe that $[I, L] = \text{span}\left[[x,y] \,\middle|\, \begin{matrix} x \in I \\ y \in L \end{matrix}\right]$ is also an ideal of $L$ since if $x \in L$, $y \in I$, $z \in L$ then

$$[\underset{\underset{L}{\cap}}{x}, \underbrace{[y,z]}_{\substack{\text{generic elem} \\ \text{of } [I,L]}}] \underset{\text{by def}}{=} \text{ad}\, x[y,z] \underset{\substack{\text{by} \\ \text{Jacobi} \\ \text{identity}}}{=} [\underbrace{[x,y]}_{\in I}, \underbrace{[x,z]}_{\in L}] \in [I,L]$$

If $[I, L] = 0$ then $I \subseteq Z(L) \overset{\text{since } L \text{ semisimple}}{=} 0$

As $I \neq 0$ is simple, we must have $I = [I, L]$

But $[I, L] = \bigoplus_j \underbrace{[I, L_j]}_{\text{this is ideal of } I} = I$ means that $[I, L_j]$ must be zero for all but one $j$ and $I = L_j$. $\square$

Our original definition of <u>semisimple</u> was the property of having no nonzero solvable ideals. Now we have a more intuitive characterization:

<u>Cor</u> $L$ is semisimple if and only $L$ is a direct sum of simple Lie algebras.

<u>Pf</u> "only if" direction: previous theorem

"if" direction: if $L = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ with $L_i$ simple

then the radical of the Killing form $\kappa$ of $L$ is

$$\bigoplus_{i=1}^{n} \mathrm{Rad}(\underbrace{\kappa|_{L_i \times L_i}}_{= \text{killing form of } L_i}) \quad \text{since } L_i^{\perp} = \bigoplus_{j \neq i} L_j.$$

But each simple $L_i$ is semisimple so $\mathrm{Rad}(\kappa|_{L_i \times L_i}) = 0$

so this direct sum is zero

$\square$

Cor If $L$ is semisimple then $L=[L,L]$, and all ideals and homomorphic images of $L$ are also semisimple.

Pf If $L=\bigoplus_i L_i$, each $L_i$ simple, then $[L_i,L_i]=L_i$ $\forall i$ and $[L_i,L_j]=0$ $\forall i\neq j$

so $[L,L]=\bigoplus_{i,j}[L_i,L_j]=\bigoplus_i L_i=L$.

If $I\leq L$ is an ideal then $I$ is semisimple as any of its solvable ideals are also ideals of $L$.

Final claim about homomorphic images: exercise. $\square$