MATH 5143 - Lecture 10

Application of Weyl's thm to abstract Jordan decomposition

Any (square) matrix / linear map $X \in \mathfrak{gl}(V)$ has a unique **Jordan decomposition** $X = X_s + X_n$ where $X_s X_n = X_n X_s$

and $X_s \in \mathfrak{gl}(V)$ is **semisimple** $\equiv$ **diagonalizable** and

$X_n \in \mathfrak{gl}(V)$ is **nilpotent** $\equiv$ strictly upper-Δ in some basis

Thm Assume L is semisimple. Then each $X \in L$ has a unique **(abstract) Jordan decomposition** $X = X_s + X_n$

where $X_s, X_n \in L$ and for any repn $\phi : L \to \mathfrak{gl}(V)$

$\phi(X) = \phi(X_s) + \phi(X_n)$ is the usual Jordan decomposition.

Recall: **faithful** $\equiv$ injective $\equiv$ $\ker \phi = 0$

Root space decomposition

⇝ this will generalize the weight space decomp we just saw for $\mathfrak{sl}_2$-modules

Suppose L is any (finite-dim nonzero) semisimple Lie algebra.

A subalgebra $T \subseteq L$ is <u>toral</u> if all of its elements are semisimple (that is, if $x \in T$ has abstract Jordan decomposition $x = x_s + x_n$ then $x = x_s$, $x_n = 0$)

[Equivalently: $x \in T$ iff L has a basis consisting of eigenvectors for $\operatorname{ad} x$]

<u>Lemma</u> Any toral subalgebra $T \subseteq L$ is abelian: $[x, y] = 0 \ \forall x, y \in T$

<u>Pf</u> Assume $T \subseteq L$ is toral. Suffices to show $\operatorname{ad}_T x \overset{\text{def}}{=} (\operatorname{ad} x)|_T$ is zero $\forall x \in T$. Since $\operatorname{ad} x$ is diagonalizable, preserves T, and $\mathbb{F}$ is alg. closed, we just can prove that $\operatorname{ad}_T x$ has no nonzero eigenvalues. (Since it is easy exercise to show that T is spanned by eigenvectors for $\operatorname{ad}_T x$)

Arguing by contradiction, suppose $[X,Y] = aY$ for some $Y \in T$, $0 \neq a \in \mathbb{F}$.

Then $(\mathrm{ad}_T Y)(X) = [Y,X] = -[X,Y] = -aY \neq 0$ is an eigenvector for $\mathrm{ad}_T Y$ with eigenvalue zero

(since $(\mathrm{ad}_T Y)(-aY) = -a[Y,Y] = 0$)

But X is a linear combination of eigenvectors for $\mathrm{ad}\, Y$ (since $Y \in T$) and also for $\mathrm{ad}_T Y$ (since $X, Y \in T$)

If we write $X = \sum_{\lambda \in \mathbb{F}} \overbrace{c_\lambda}^{\text{some coeff in } \mathbb{F}} \underbrace{X_\lambda}_{\text{eigenvectors for ad } Y \text{, which are linearly independent}}$ where $[Y, X_\lambda] = \lambda X_\lambda$ then

eigenvalues for $\mathrm{ad}\, Y$

$0 \neq [Y,X] = \sum_{\lambda \neq 0} c_\lambda \lambda X_\lambda$ and $[Y,[Y,X]] = \sum_{\lambda \neq 0} c_\lambda \lambda^2 X_\lambda \neq 0$

contradicting $[Y,[Y,X]] = -a[Y,Y] = 0$. Thus $[X,Y] = 0$ $\forall X, Y \in T$. $\square$

Choose a maximal toral subalgebra $H \subseteq L$.

not necessarily unique choice

(there is no other toral subalgebra $T \subseteq L$ with $H \subsetneq T$)

Ex If $L = \mathfrak{sl}_n(\mathbb{F})$ then one choice for H is the subalgebra of (traceless) diagonal matrices.

L is never abelian (since L is semisimple, $Z(L) = 0$) so any toral subalgebra $T \subseteq L$ has $T \neq L$, and is never an ideal.

H (our maximal toral subalgebra) is abelian so $\mathrm{ad}_L H$ is a family of commuting diagonalizable / semisimple linear maps $L \to L$. Therefore $\mathrm{ad}_L H$ is simultaneously diagonalizable, meaning there is a decomposition

$$L = \bigoplus_{\alpha \in H^*} L_\alpha$$

if $\alpha = 0$ then $L_0 = \{x \in L \mid [x,h] = 0 \ \forall h \in H\} = C_L(H)$

could be zero, must be zero for all but finitely many α

where $H^* = $ (linear maps $H \to \mathbb{F}$) and

$$L_\alpha = \{x \in L \mid [h,x] = \alpha(h)x \text{ for all } h \in H\}$$

① If $L_\alpha \neq 0$ and $0 \neq \alpha \in H^*$ then α is called a <u>root</u>
Let Φ be set of all roots: this is a finite subset of $H^* \setminus 0$.

② We call $L = \underbrace{C_L(H)}_{= L_0} \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ a Cartan / root space decomposition of L.

We have $L = C_L(H) \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$, $L_\alpha = \{x \in L \mid [h,x] = \alpha(h)x \ \forall h \in H\}$

Prop For all $\alpha, \beta \in H^*$, $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$

Pf Jacobi identity for $h \in H$, $X \in L_\alpha$, $Y \in L_\beta$ implies that

$$[h,[X,Y]] = -[X,[Y,h]] - [Y,[h,X]]$$

$$= [X,[h,Y]] + [[h,X],Y] = (\alpha(h)+\beta(h))[X,Y] \square$$

so $[X,Y] \in L_{\alpha+\beta}$

Prop If $X \in L_\alpha$ and $0 \neq \alpha \in H^*$ then $\mathrm{ad}\,X$ is nilpotent

Pf $(\mathrm{ad}\,X)^n(Y) \in L_{n\alpha+\beta}$ for any $Y \in L_\beta$, and if $n >> 0$ then $L_{n\alpha+\beta} = 0$ since $\dim L < \infty$. Since $L = \bigoplus_{\beta \in H^*} L_\beta$ it follows that $(\mathrm{ad}\,X)^n = 0$ for $n >> 0$. $\square$

Prop If $\alpha, \beta \in H^*$ with $\alpha+\beta \neq 0$ and $\kappa(X,Y) \stackrel{\text{def}}{=} \operatorname{trace}(\operatorname{ad}X \operatorname{ad}Y)$ denotes the Killing form of L (which is nondegenerate) then $\kappa(X,Y) = 0 \ \forall X \in L_\alpha, Y \in L_\beta$. Thus L_α and L_β are orthogonal wrt κ if $\alpha+\beta \neq 0$.

Pf Since $\alpha+\beta \neq 0$, there is $h \in H$ with $(\alpha+\beta)(h) \neq 0$.

Let $X \in L_\alpha$, $Y \in L_\beta$. Then

by associativity of κ

$$\kappa([h,X],Y) = -\kappa([X,h],Y) \stackrel{\downarrow}{=} -\kappa(X,[h,Y])$$

$$\alpha(h)\kappa(X,Y) = -\beta(h)\kappa(X,Y)$$

$$\Rightarrow (\alpha+\beta)(h)\kappa(X,Y) = 0 \Rightarrow \kappa(X,Y) = 0. \ \square$$

<u>Cor</u> Killing form κ of L restricts to a nondegenerate form on $L_0 = C_L(H) = \{x \in L \mid [x,h]=0 \ \forall h \in H\}$

<u>Pf</u> Let $0 \neq x \in L_0$.

Since $\kappa(x,y)=0$ for all $y \in \bigoplus_{\alpha \in \Phi} L_\alpha$ by prev. prop., we must have $\kappa(x,y) \neq 0$ for some $y \in L_0$ since otherwise $\kappa: L \times L \to \mathbb{F}$ would be degenerate with $\kappa(x,\cdot) = 0 \in L^*$. $\square$

Easy fact from linear algebra: If $X, Y \in \mathfrak{gl}(V)$ with $\dim V < \infty$ and $XY = YX$ and Y is nilpotent, then XY is also nilpotent (since $(XY)^n = X^n Y^n$) and $\operatorname{trace}(XY) = \operatorname{trace}(Y) = 0$.

<u>Thm</u> Suppose H is a maximal toral subalgebra of a semisimple Lie algebra L with $\dim L < \infty$. Then

$$H = C_L(H) = \{ x \in L \mid [x, h] = 0 \ \forall h \in H \}$$

So the Cartan decomp of L wrt H is just

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

Pf. Let $C = C_L(H)$. We proceed with a series of claims:

Claim 1 If $x \in C$ then $x_s \in C$ and $x_n \in C$

the parts of the Jordan decomp $x = x_s + x_n$

Pf If $x \in C$ then $\operatorname{ad} x$ maps $H \to 0$. Since $(\operatorname{ad} x)_s$ and $(\operatorname{ad} x)_n$ are polynomials in $\operatorname{ad} x$ with zero constant term, they also map $H \to 0$. But $(\operatorname{ad} x)_s = \operatorname{ad}(x_s)$ and $(\operatorname{ad} x)_n = \operatorname{ad}(x_n)$ so this means that $x_s, x_n \in C$. $\square$

Claim 2 If $x = x_s \in C$ then $x \in H$

Pf Suppose $x = x_s \in C$. Then $H + \mathbb{F}x$ is a toral subalgebra so must be equal to H, so $x \in H$. $\square$

(by maximality of H)

Claim 3 $\kappa|_{H\times H}$ is nondegenerate [Note: we have already shown that $\kappa|_{C\times C}$ is nondegenerate]

Pf. Suppose $\kappa(h, H) = 0$ for some $h \in H$. Want to show that $h = 0$. Consider some $X \in C$. By Claims 1 and 2, we have $X_n \in C$ and $X_s \in H \subseteq C$. So $\kappa(h, X) = \kappa(h, X_n) = \operatorname{trace}(\operatorname{ad} h \operatorname{ad} X_n) = 0$

by standard linear algebra fact before thm, as $\operatorname{ad} h$ and $\operatorname{ad} X_n$ commute (exercise)

therefore $\kappa(h, C) = 0$. But this contradicts

as $H \subseteq C$, unless $h = 0$ as desired $\square$

<u>Claim 4</u> C is nilpotent, ie, $\mathrm{ad}_C x$ is nilpotent $\forall x \in C$

<u>Pf</u> If $x = x_s \in C$ then $x \in H$ so $\mathrm{ad}_C x = 0$ (which is clearly nilpotent)

If $x = x_n \in C$ then $\mathrm{ad}_C x_n$ is nilpotent by definition.

For general $x = x_s + x_n \in C$, we have $x_n, x_s \in C$ and

$\mathrm{ad}_C x_s$ commutes with $\mathrm{ad}_C x_n$, so $\mathrm{ad}\, x = \mathrm{ad}\, x_s + \mathrm{ad}\, x_n$ is nilpotent. $\square$

<u>Claim 5</u> $H \cap [C,C] = 0$

<u>Pf</u> $\kappa(H, [C,C]) \underset{\uparrow}{=} \kappa([H,C], C) = \kappa(0, C) = 0.$

by associativity of form

Since $\kappa|_{H \times H}$ is nondegenerate, this means no nonzero $x \in H$ is in $[C,C]$.

$\square$

Claim 6 C is abelian, meaning $[C,C]=0$.

Pf Suppose $[C,C]\neq 0$. This is a nonzero ideal of C, which is nilpotent by claim 4. So, by theorem proved to show Engel's thm, $\operatorname{ad} C$ acts on $[C,C]$ as nilpotent linear transforms, which in some basis are all strictly upper-Δ matrices. In other words, there is some $0\neq z\in[C,C]$ with $[x,z]=0$ for all $x\in C$. This element is evidently in $[C,C]\cap Z(C)$. It cannot be semisimple as then we would have $0\neq z=z_s\in H\cap[C,C]=0$. Thus (as $z\neq 0$) we must have $0\neq z_n\in C$. But $\operatorname{ad} z_n$ is polynomial in $\operatorname{ad} z$ w/o constant term, so $z_n\in Z(C)$. But then $\kappa(z_n,C)=0$ contradicting that $\kappa|_{C\times C}$ is nondegenerate. $\square$

Finally <u>Claim 7</u> $C = H$

as C is nilpotent, all of its elements are ad-nilpotent

<u>Pf</u> If $C \neq H$ then there exists a nilpotent nonzero element $0 \neq x = x_n \in C$. But as $Z(C) = C$ by claim 6, the argument just given implies that

ad x nilpotent, commutes with all ad y for $y \in C$, since C abelian

$$\kappa(x, C) = 0$$

contradicting $\kappa|_{C \times C}$ is nondegenerate. $\square$