Math 5143 - Lectures 11 + 12

Last time: $\mathfrak{sl}_2(\mathbb{F}) = \mathbb{F}\text{-span}\left\{ h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

($\mathbb{F}$ alg. closed field of char. zero)

Classification thm for finite dim. irreducible $\mathfrak{sl}_2(\mathbb{F})$-reprs

For each integer $m \geq 0$ there is a unique isomorphism class of irreducible $\mathfrak{sl}_2(\mathbb{F})$-modules V with $\dim V = m+1$.

This irreducible module has a basis $v_0\, v_1\, v_2 \ldots v_m$ such that

$$h v_i = (m-2i) v_i$$
$$y v_i = (i+1) v_{i+1} \quad (v_{m+1} = 0)$$
$$x v_i = (m-i+1) v_{i-1} \quad (v_{-1} = 0$$

(of weight m)

The vector v_0 (or any nonzero scalar multiple) is a <u>highest weight vector</u>

weight vectors $\Leftrightarrow$ eigenvectors for h, weights $\Leftrightarrow$ eigenvalues for h

Important picture of irred. $\mathfrak{sl}_2(\mathbb{F})$-module:

$$V = V_m \oplus V_{m-2} \oplus \cdots \oplus V_{-m+2} \oplus V_{-m}$$

(where $V_{m-2i} \overset{\text{def}}{=} \mathbb{F} v_i$)

$$\begin{tikzcd} V_m \arrow[r, shift left, "y"] & V_{m-2} \arrow[l, shift left, "x"] \arrow[r, shift left, "y"] & V_{m-4} \arrow[l, shift left, "x"] \arrow[r, shift left, "y"] & \cdots \arrow[l, shift left, "x"] \arrow[r, shift left, "x"] & V_{-m} \arrow[l, shift left, "x"] \end{tikzcd}$$

each subspace is eigenspace for h with eigenvalues $m, m-2, \ldots, -m$

Root space decomposition

Let L be a nonzero, finite dim. semisimple Lie alg.

A subalgebra $H \subseteq L$ is <u>toral</u> if every $x \in H$ has $x = x_s$ and $x_n = 0$, where $x = x_s + x_n$ is the abstract Jordan decomp. of x: $x_s, x_n \in L$ are the unique elems with

$$\begin{cases} x = x_s + x_n \\ \operatorname{ad} x_s \text{ diagonalizable } L \to L \\ \operatorname{ad} x_n \text{ nilpotent} \\ [\operatorname{ad} x_s, \operatorname{ad} x_n] = 0 \end{cases}$$

I.e., every element of H is semisimple.

Thm (a) Any toral subalgebra H is abelian ($[H,H]=0$)

(b) Any maximal toral subalgebra H is self-centralizing

$$\left(H = C_L(H) \overset{\text{def}}{=} \{ x \in L \mid [x,h]=0 \ \forall h \in H \} \right)$$

(c) The killing form of L, given by $\kappa(x,y) = \operatorname{trace}(\operatorname{ad} x \operatorname{ad} y)$, restricts to a nondegenerate form on H if H is maximal toral subalgebra

Given a maximal toral subalgebra $H \leq L$, the corresponding root space decomposition is $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ where

Φ is a finite subset of H^*, $L_\alpha \overset{\text{def}}{=} \{ x \in L \mid [h,x] = \alpha(h) x \ \forall h \in H \}$

$0 \notin \Phi$ since $H = L_0$. Existence of this decomp. is consequence of (a)

Call L_α a root space and $\alpha \in \Phi$ a root

Ex Suppose $L = \mathfrak{sl}_3(\mathbb{F}) = 3\times 3$ traceless matrices

For a maximal toral subalgebra, take $H = \left\{ \begin{bmatrix} a&0&0\\0&b&0\\0&0&c \end{bmatrix} \;\middle|\; a+b+c=0 \right\}$

Let $\varepsilon_i : H \to \mathbb{F}$ by $\varepsilon_i(M) = M_{ii}$ (diagonal entry in row i)

Each $\varepsilon_i \in H^*$, but $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are not a basis as $\dim H^* = 2$.

Then we have

$\begin{bmatrix}a&&\\&b&\\&&c\end{bmatrix}\begin{bmatrix}\cdot&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{bmatrix} - \begin{bmatrix}\cdot&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{bmatrix}\begin{bmatrix}a&&\\&b&\\&&c\end{bmatrix} = (a-b)\begin{bmatrix}\cdot&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{bmatrix} = (\varepsilon_1-\varepsilon_2)\left(\begin{bmatrix}a&&\\&b&\\&&c\end{bmatrix}\right)\begin{bmatrix}\cdot&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{bmatrix}$

$$L = H \oplus \underbrace{\mathbb{F}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}}_{=L_{\varepsilon_1-\varepsilon_2}} \oplus \underbrace{\mathbb{F}\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}}_{=L_{\varepsilon_2-\varepsilon_1}} \oplus \underbrace{\mathbb{F}\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}}_{=L_{\varepsilon_1-\varepsilon_3}} \oplus \underbrace{\mathbb{F}\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}}_{=L_{\varepsilon_3-\varepsilon_1}}$$

$$\oplus \underbrace{\mathbb{F}\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}}_{=L_{\varepsilon_2-\varepsilon_3}} \oplus \underbrace{\mathbb{F}\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}}_{=L_{\varepsilon_3-\varepsilon_2}}$$

L_α is defined for any $\alpha \in H^*$ but may be zero

We have $\Phi = \{ \varepsilon_i - \varepsilon_j \mid 1 \le i,j \le 3,\ i \ne j \}$

Recall we have a root space decomp

$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ for finite set $\Phi \subset H^* \setminus 0$

with $L_\alpha = \{ x \in L \mid [h, x] = \alpha(h) x \ \forall h \in H \}$

Because $\kappa|_{H \times H}$ is nondegenerate, for each $\alpha \in H^*$, there is a unique element $t_\alpha \in H$ such that $\alpha(h) = \kappa(t_\alpha, h) \ \forall h \in H$.

Orthogonality properties of Φ (this set is det'd by H)

ⓐ $H^* = \mathbb{F}\text{-span}\{\alpha \in \Phi\} = \mathbb{F}\Phi$ → otherwise, there is some $0 \neq h \in H$ with $\alpha(h) = 0 \ \forall \alpha \in \Phi$ and then $[h, L_\alpha] = \alpha(h) L_\alpha = 0 \ \forall \alpha \in \Phi$ and also $[h, H] = 0 \Rightarrow [h, L] = 0 \Rightarrow 0 \neq h \in Z(L) = 0$

[contradiction]

ⓑ If $\alpha \in \Phi$ then $-\alpha \in \Phi$

By a prop last time, $\kappa(L_\alpha, L_\beta) = 0$ if $\alpha, \beta \in \Phi$ and $\alpha + \beta \neq 0$, and $\kappa(L_\alpha, H) = 0$. So if $\alpha \in \Phi$, $-\alpha \notin \Phi$ then it would follow that $\kappa(L_\alpha, L) = 0$, contradicting nondegeneracy of κ.

ⓒ If $\alpha \in \Phi$, $x \in L_\alpha$, $y \in L_{-\alpha}$ then $[x, y] = \kappa(x, y) t_\alpha$

If $h \in H$ then $\kappa(h, [x, y]) = \kappa([h, x], y) = \alpha(h)\kappa(x, y) = \kappa(t_\alpha, h)\kappa(x, y)$
$= \kappa(h, \kappa(x, y) t_\alpha) \Rightarrow \kappa(h, [x, y] - \kappa(x, y) t_\alpha) = 0 \ \forall h \in H \Rightarrow [x, y] = \kappa(x, y) t_\alpha$ by non degeneracy of $\kappa|_{H \times H}$

Ex If $L = sl_3(\mathbb{F})$ where every root has form $\alpha = \varepsilon_i - \varepsilon_j$ $(i \neq j)$

and every root space is $L_{\varepsilon_i - \varepsilon_j} = \mathbb{F}E_{ij}$ ← matrix with 1 in position (i,j), 0 elsewhere

it follows that $t_{\varepsilon_i - \varepsilon_j} = \dfrac{1}{\kappa(E_{ij}, E_{ji})}[E_{ij}, E_{ji}] = \underbrace{\dfrac{1}{4}}_{\text{follows from computations like in HW2}}(E_{ii} - E_{jj})$

This works if $L = sl_n(\mathbb{F})$ for any n.

More properties of root space decomposition:

(d) If $\alpha \in \Phi$ then $[L_\alpha, L_{-\alpha}] = \mathbb{F}\text{-span}\{t_\alpha\} \neq 0$

Just need to show $[L_\alpha, L_{-\alpha}] \neq 0$ given (c). If $0 \neq x \in L_\alpha$ and $\kappa(x, L_{-\alpha}) = 0$ then $\kappa(x, L) = 0$, which is impossible as κ is nondegenerate

(e) $\alpha(t_\alpha)$ (which by definition is $\kappa(t_\alpha, t_\alpha)$) is nonzero for all $\alpha \in \Phi$

$\hookrightarrow$ [$\alpha(h) = \kappa(t_\alpha, h)$ $\forall h \in H$]

Pf We can find $x \in L_\alpha$, $y \in L_{-\alpha}$ with $[x, y] = t_\alpha$ by (d)

If $\alpha(t_\alpha) = 0$ then $\underbrace{[t_\alpha, x]}_{=\alpha(t_\alpha)x} = \underbrace{[t_\alpha, y]}_{(-\alpha)(t_\alpha)y} = 0$.

In this case $\operatorname{ad} t_\alpha$ is nilpotent and semisimple (as $t_\alpha \in H$) so $\operatorname{ad} t_\alpha = 0$

$\Rightarrow 0 \neq t_\alpha \in Z(L) = 0$

a contradiction

$\operatorname{ad} x$, $\operatorname{ad} y$, $\operatorname{ad} t_\alpha$

generate a solvable subalgebra of $\mathfrak{gl}(L)$

so by Lie's thm there is a basis for L with

$\operatorname{ad} x$ and $\operatorname{ad} y$ upper-Δ, and then

$\operatorname{ad} t_\alpha = \operatorname{ad}[x, y] = [\operatorname{ad} x, \operatorname{ad} y]$ is strictly upper-Δ.

(f) If $\alpha \in \Phi$ and $X_\alpha \in L_\alpha$ is nonzero then there is some $Y_\alpha \in L_{-\alpha}$ such that $\mathbb{F}\text{-span}\{X_\alpha, Y_\alpha, H_\alpha\} \cong sl_2(\mathbb{F})$

via the obvious map $X_\alpha \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$Y_\alpha \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$H_\alpha \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

where $H_\alpha \overset{\text{def}}{=} [X_\alpha, Y_\alpha]$

Pf Define Y_α such that $\kappa(X_\alpha, Y_\alpha) = \frac{2}{\kappa(t_\alpha, t_\alpha)}$ and then do some checking. $\square$

(g) In setup of (f) the element $H_\alpha \overset{\text{def}}{=} [X_\alpha, Y_\alpha]$ has $H_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} = -H_{-\alpha}$.

Pf is straightforward $\square$

We continue our setup: $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ $\begin{cases} L_\alpha = \{x \mid [h,x] = \alpha(h)x \ \forall h \in H\} \\ \Phi \subset H^* \setminus 0 \\ H \text{ maximal toral} \\ \alpha \in H^* \text{ belongs to } \Phi \text{ iff} \\ (0 \neq \alpha, L_\alpha \neq 0) \end{cases}$

Integrality properties of Φ

<u>Prop</u> (a) $\dim L_\alpha = 1 \quad \forall \alpha \in \Phi$

(b) If $\alpha \in \Phi$ then $\mathbb{F}\alpha \cap \Phi = \{-\alpha, \alpha\}$

<u>Pf</u> Let $\alpha \in \Phi$. Let $S_\alpha = \mathbb{F}\text{span}\{x_\alpha, y_\alpha, H_\alpha = [x_\alpha, y_\alpha]\} \cong \mathfrak{sl}_2(\mathbb{F})$

where $0 \neq x_\alpha \in L_\alpha$, $0 \neq y_\alpha \in L_{-\alpha}$.

Let $M = \bigoplus_c L_{c\alpha} \oplus H = H \oplus L_\alpha \oplus L_{-\alpha} \oplus$ (possibly other root spaces)

$(0 \neq c \in \mathbb{F} \text{ and } c\alpha \in \Phi)$ but we will prove this is zero

M is an S_α-module with weights are 0 and $2c$ since
(eigenvalues of H_α)

$$c\alpha(H_\alpha) = c \cdot \alpha\left(\frac{2t_\alpha}{\alpha(t_\alpha)}\right) = 2c \text{ by previous props.}$$

$\Rightarrow$ we must have $c \in \frac{1}{2}\mathbb{Z}$ since all $\mathfrak{sl}_2$-weights are integers.

Every irreducible S_α-submodule of M of even highest weight contributes one dimension to the zero weight space of M

(which is just H)

0-eigenspace of $H_\alpha \in S_\alpha$

But $S_\alpha \subseteq M$ is irreducible and

$$H = \underbrace{\ker(\alpha)} \oplus \mathbb{F} H_\alpha$$

0-weight space in S_α

S_α acts as $\boxed{\text{zero}}$ on this subspace, which has $\boxed{\dim = \dim H - 1}$

Since we already have $L_\alpha, L_{-\alpha} \subseteq S_\alpha$ it must hold that $L_{c\alpha} = 0$ if c is an even integer with $c \neq -2, 0, 2$

Conclude that $\alpha \in \Phi$ then $2\alpha \notin \Phi$. Hence we cannot have $\alpha, \frac{1}{2}\alpha \in \Phi$ so if $\alpha \in \Phi$ then $\frac{1}{2}\alpha \notin \Phi$. This means that 1 cannot occur as a weight for $M \implies M = H + S_\alpha = \ker(\alpha) \oplus \mathbb{F}H_\alpha \oplus \mathbb{F}X_\alpha \oplus \mathbb{F}Y_\alpha$

so $\dim L_\alpha = 1$ $\square$