Math 5143 - Lecture 15

"Simple roots" and the Weyl group

Φ is a root system in vectorspace E with Weyl group W

A **base** or **simple system** for Φ is a basis Δ for E such that each $\alpha \in \Phi$ can be written as $\alpha = \sum_{\beta \in \Delta} k_{\alpha\beta} \beta$ where coefficients $k_{\alpha\beta}$ are either (1) all nonnegative integers or (2) all nonpositive integers.

Necessarily $|\Delta| = \dim E$. Not clear apriori that any base exists.

Ex In each root system in $\mathbb{R}^2$, the roots labeled $\{\alpha, \beta\}$ form a base.

Lemma If Δ is a base of Φ and $\alpha, \beta \in \Delta$ have $\alpha \neq \beta$, then $\alpha - \beta \notin \Phi$ so $(\alpha, \beta) \leq 0$.

Pf If $(\alpha, \beta) > 0$ then our earlier lemma says $\alpha - \beta \in \Phi$ since if $\alpha \neq \beta$ then also $\alpha \neq -\beta$ (since elems of Δ are linearly independent)

But if $\alpha - \beta \in \Phi$ then Δ would not be a base. $\square$

Given a simple system Δ for Φ, define the height of a root

$\alpha = \sum_{\beta \in \Delta} k_{\alpha\beta} \beta$ to be the sum $\boxed{ht(\alpha) = \sum_{\beta \in \Delta} k_{\alpha\beta} \in \mathbb{Z} \setminus 0.}$

We also define $\Phi^+ = \{\alpha \in \Phi \mid ht(\alpha) > 0\}$ and $\Phi^- = -\Phi^+$

so that $\Phi = \Phi^+ \sqcup \Phi^-$. Call Φ^+ the set of positive roots,
Φ^- the set of negative roots.

Thm Φ does have a base/simple system.

For each $\gamma \in E$ define $\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\}$.

One can always choose $\gamma \in E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$ and we call such γ regular.

If γ is regular then $\Phi = \Phi^+(\gamma) \sqcup \Phi^-(\gamma)$ where $\Phi^-(\gamma) = -\Phi^+(\gamma)$.

Call $\alpha \in \Phi^+(\gamma)$ indecomposable if we cannot write $\alpha = \beta_1 + \beta_2$ where $\beta_i \in \Phi^+(\gamma)$.

Thm If $\gamma \in E$ is regular, then the set $\Delta(\gamma)$ of indecomposable roots in Φ
is a base, and every base arises in this way.

Pf We make a series of claims.

defined to be the roots in $\Phi^+(\gamma)$ that are indecomposable

① Each $\alpha \in \Phi^+(\gamma)$ is in $\mathbb{Z}_{\geq 0}$-span$\{\beta \in \Delta(\gamma)\}$

Pf Otherwise, choose $\alpha \in \Phi^+(\gamma)$ not in with (α, γ) minimal.

Then $\alpha = \beta_1 + \beta_2$ for some $\beta_1, \beta_2 \in \Phi^+(\gamma)$ [α cannot be indecomposable]

Thus $(\alpha, \gamma) = \underbrace{(\beta_1, \gamma)}_{>0} + \underbrace{(\beta_2, \gamma)}_{>0}$ so by minimality of (α, γ)

it must hold that $\beta_1, \beta_2 \in \mathbb{Z}_{\geq 0}$-span$\{\beta \in \Delta(\gamma)\}$, a contradiction

(as α is not in) $\square$

② If $\alpha, \beta \in \Delta(\gamma)$ and $\alpha \neq \beta$, then $(\alpha, \beta) \leq 0$.

Pf Otherwise $\alpha - \beta \in \Phi$, $\beta \neq \pm\alpha$, so $\alpha - \beta$ or $\beta - \alpha$ is in $\Phi^+(\gamma)$

But then $\alpha = \beta + (\alpha - \beta)$ or $\beta = \alpha + (\beta - \alpha)$ would be decomposable. $\square$

③ $\Delta(\gamma)$ is linearly independent

Pf Suppose we can write $0 = \sum_\alpha c_\alpha \alpha - \sum_\beta d_\beta \beta$

where α, β range over disjoint subsets of $\Delta(\gamma)$ and $c_\alpha, d_\beta \geq 0$

Then $0 \leq \left(\sum_\alpha c_\alpha \alpha, \sum_\alpha c_\alpha \alpha \right) = \left(\sum_\alpha c_\alpha \alpha, \sum_\beta d_\beta \beta \right)$

$$= \sum_{\alpha, \beta} \underbrace{c_\alpha d_\beta}_{\geq 0} \underbrace{(\alpha, \beta)}_{\leq 0 \text{ by previous claim}} \leq 0$$

$\Rightarrow$ so all $c_\alpha = 0$. Similarly derive that all $d_\beta = 0$. $\square$

④ $\Delta(\gamma)$ is a base of Φ.

Pf clear from ①②③

⑤ Every base of Φ arises as $\Delta(\gamma)$ for some regular $\gamma \in E$.

Pf Given some base Δ for Φ, we need to find γ with $\Delta = \Delta(\gamma)$.

Choose a regular γ with $(\gamma, \alpha) > 0$ for all $\alpha \in \Delta$ [It's a (HW) exercise to show we can always do this]. Then $\Phi^{+/-} = \Phi^{+/-}(\gamma)$ so every $\alpha \in \Delta$ must be indecomposable wrt γ. This means $\Delta \subseteq \Delta(\gamma)$. As $|\Delta| = |\Delta(\gamma)| = \dim E$, must have $\Delta = \Delta(\gamma)$.

$\square$

Call elems of Δ <u>simple roots</u>

The hyperplanes H_α for $\alpha \in \Phi$ divide E into finitely many regions. We call the connected components of

$$E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$$ the <u>Weyl chambers</u> of E.

Properties of simple roots

Fix a base Δ of Φ and define $\Phi^{+/-}$ relative to Δ. Elems of Φ^+ are positive roots, elems of Φ^- are negative roots.

Lemma If $\alpha \in \Phi^+$ but $\alpha \notin \Delta$ then $\alpha - \beta \in \Phi^+$ for some $\beta \in \Delta$.

Pf If $(\alpha, \beta) \leq 0$ for all $\beta \in \Delta$ then argument in proof of 3 in previous proof would show that $\Delta \sqcup \{\alpha\}$ is linearly independent. As this is impossible, must have $(\alpha, \beta) > 0$ for some $\beta \in \Delta$ and then $\alpha - \beta \in \Phi$. Since α, β cannot be proportional, $\alpha - \beta$ must be in Φ^+ (since at least one coeff in $\alpha - \beta = \sum_{\delta \in \Delta} c_\delta \delta$ must have $c_\delta > 0$). $\square$

By induction: **Cor** Each $\alpha \in \Phi^+$ can be written $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_k$ where $\alpha_i \in \Delta \; \forall i$ and where each partial sum $\alpha_1 + \alpha_2 + \dots + \alpha_j \in \Phi^+$ for $1 \leq j \leq k$.

Lemma If $\alpha \in \Delta$ then $\underbrace{r_\alpha(\alpha) = -\alpha}_{\text{holds by def, for any } 0 \neq \alpha \in E}$ and $\underbrace{r_\alpha(\Phi^+ \setminus \{\alpha\}) = \Phi^+ \setminus \{\alpha\}}_{\text{nontrivial}}$

Pf Suppose $\beta \in \Phi^+ \setminus \{\alpha\}$. Write $\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$ where $k_\gamma \in \mathbb{Z}_{\geq 0}$.

Note: β is not proportional to α. Thus $k_\gamma \neq 0$ for some $\gamma \neq \alpha$.

Then the coeff of γ in $r_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ is also $k_\gamma > 0$,

so $r_\alpha(\beta)$ must still be in Φ^+ since it is a valid root. $\square$

(lemma now follows as $r_\alpha : E \to E$ is a bijection)

Cor Set $\delta = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$ then $r_\alpha(\delta) = \delta - \alpha \;\; \forall \alpha \in \Delta$.

<u>Lemma</u> Suppose we have a sequence $\alpha_1, \alpha_2, \dots, \alpha_m \in \Delta$. Write $r_i = r_{\alpha_i}$.

Suppose $r_1 r_2 \cdots r_{m-1}(\alpha_m) \in \Phi^-$. Then $r_1 r_2 \cdots r_m = r_1 \cdots r_{s-1} r_{s+1} \cdots r_{m-1}$ for some index $1 \le s \le m-1$. [The roots $\alpha_1, \alpha_2, \dots, \alpha_m$ don't need to be all distinct]

<u>Pf</u>. Set $\beta_i \overset{\text{def}}{=} r_{i+1} r_{i+2} \cdots r_{m-1}(\alpha_m)$, with $\beta_{m-1} \overset{\text{def}}{=} \alpha_m$.

Then $\beta_0 \in \Phi^-$ and $\beta_{m-1} \in \Delta \subset \Phi^+$ so there is some smallest index s with $\beta_s \in \Phi^+$. Then $r_s(\beta_{s-1}) = \beta_s$ since $r_s^2 = 1 \Rightarrow r_s(\beta_s) = \beta_{s-1} \in \Phi^- \Rightarrow \beta_s = \alpha_s$ by prev. lem.

$\Rightarrow r_s \overset{\text{def}}{=} r_{\alpha_s} = r_{\beta_s} = r_{r_{s+1} r_{s+2} \cdots r_{m-1}(\alpha_m)} = (r_{s+1} \cdots r_{m-1}) r_m (r_{m-1} \cdots r_{s+1})$

[since $\sigma r_\alpha \sigma^{-1} = r_{\sigma(\alpha)}$]

Result follows by substituting this expr for r_s, noting that $r_i^2 = 1$. $\square$

↳ into $r_1 \cdots r_s \cdots r_m$

Cor. If $\sigma = r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_m}$ is an expression for $\sigma \in W$ with m as small as possible and $\alpha_i \in \Delta$, then $\sigma(\alpha_m) \in \Phi^-$.

Recall: Φ is a root system with Weyl group W.

Prop Any given $\alpha \in \Phi$ belongs to some base of Φ.

Pf The hyperplanes H_β for $\beta \in \Phi \setminus [\pm\alpha]$ are distinct from H_α, so if we choose $\gamma \in H_\alpha$ with $\gamma \notin H_\beta$ $\forall \beta \in \Phi \setminus [\pm\alpha]$, and then choose some regular γ' close to γ with $(\gamma', \alpha) = \varepsilon > 0$ and $(\gamma', \beta) > \varepsilon$ $\forall \beta \in \Phi \setminus [\pm\alpha]$ then we'll have $\alpha \in \Delta(\gamma')$. $\square$

Fix a base Δ for Φ.

Thm If Δ' is any base for Φ then there exists a unique element $\sigma \in W$ with $\sigma(\Delta') = \Delta$. Moreover, it holds that $W = \langle r_\alpha \mid \alpha \in \Delta \rangle$ [Recall: $W \overset{\text{def}}{=} \langle r_\alpha \mid \alpha \in \Phi \rangle$]

Pf Let $\tilde{W} = \langle r_\alpha \mid \alpha \in \Delta \rangle \subseteq W$. We'll show below that $\tilde{W} = W$.

Let $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ and choose a regular $\gamma \in E$ along with $\sigma \in \tilde{W}$ such that $(\sigma(\gamma), \delta)$ is maximal. If α is simple root then $r_\alpha \sigma \in \tilde{W}$ so our maximality assumption $\Rightarrow$ $(\sigma(\gamma), \delta) \geq (r_\alpha \sigma(\gamma), \delta)$

$= (\sigma(\gamma), r_\alpha(\delta)) = (\sigma(\gamma), \delta - \alpha) = (\sigma(\gamma), \delta) - (\sigma(\gamma), \alpha) \ \forall \alpha \in \Delta$

$\uparrow$

$(r_\alpha(x), y) = (x, r_\alpha(y)) \ \forall x, y \in E, \alpha \in \Phi$ (check this by comparing formulas)

Thus $(\sigma(\gamma), \alpha) \geq 0 \ \forall \alpha \in \Delta$. Equality never holds since γ is regular and

$$0 \neq (\gamma, \sigma^{-1}(\alpha)) = (\sigma(\gamma), \alpha)$$

Thus we have $(\sigma(\gamma), \alpha) > 0 \ \forall \alpha \in \Delta$.

If Δ' is any base then $\Delta' = \Delta(\gamma)$ for some regular $\gamma \in E$ and if we choose $\sigma \in \tilde{W}$ as above then evidently $\Delta = \Delta(\sigma(\gamma)) = \sigma^{-1}(\Delta(\gamma)) = \sigma^{-1}(\Delta')$.

So for any base Δ' there is at least some $\sigma \in \tilde{W} \subseteq W$ with $\sigma(\Delta') = \Delta$.

To show that $\tilde{W} = W$, it suffices to check that $r_\alpha \in \tilde{W} \ \forall \alpha \in \Phi$. Given $\alpha \in \Phi$, choose a base Δ' with $\alpha \in \Delta'$, and then choose $\sigma \in \tilde{W}$ with $\sigma(\Delta') = \Delta$. Set $\beta = \sigma(\alpha) \in \Delta$, and then we have

$$r_\beta = r_{\sigma(\alpha)} = \sigma r_\alpha \sigma^{-1} \in \tilde{W} \quad \text{so} \quad r_\alpha = \sigma^{-1} r_\beta \sigma \in \tilde{W} \text{ as well.}$$

because $\beta \in \Delta$, so $s_\beta \in \tilde{W}$

Finally, need to show that the element $\sigma \in \tilde{W} = W$ with $\sigma(\Delta') = \Delta$ is unique for a given base Δ' of Φ.

We appeal to technical lemma above: it's enough to show that if $\sigma \in W$ has $\sigma(\Delta) = \Delta$ then $\sigma = 1$.

Assume $\sigma(\Delta) = \Delta$ and write $\sigma = r_1 r_2 \cdots r_m$ where $r_i = r_{\alpha_i}$ for some simple roots $\alpha_1, \alpha_2, \dots, \alpha_m \in \Delta$, and assume m is minimal. If $\sigma \neq 1$ then $m > 0$ so by corollary above $\sigma(\alpha_m) \in \Phi^- \Rightarrow \sigma(\Delta) \neq \Delta \subseteq \Phi^+$

Thus the only way to have $\sigma(\Delta) = \Delta$ is if $m = 0$ and then $\sigma = 1$ $\square$

Fix an ordering $\alpha_1\ \alpha_2\ \alpha_3 \cdots \alpha_n$ of the roots in Δ.

[Here $\Delta = \{\alpha_1, \dots, \alpha_n\}$ and $\alpha_i \neq \alpha_j$ for $i \neq j$]

We call any minimal length expression

$$\sigma = r_{i_1} r_{i_2} \cdots r_{i_\ell} \quad \text{where } r_j \overset{\text{def}}{=} r_{\alpha_j}$$

a <u>reduced expression</u> for $\sigma \in W$. Set $\ell(w) = \ell$

Call this the <u>length</u> of w.

<u>Prop</u> If $\sigma \in W$ then $\ell(\sigma) = \#\{\alpha \in \Phi^+ \mid \sigma(\alpha) \in \Phi^-\}$

Note: this gives $\ell(r_\alpha) = 1 \ \forall \alpha \in \Delta$.

<u>Pf</u>. Use induction + earlier lemmas, see textbook. $\square$