MATH 5143 – Lecture 18

Math 5143 – Lecture 9

Last time: Cartan matrix, Coxeter diagrams, Dynkin diagrams $\Big\}$ Classification of irreducible root systems (and by extension all simple Lie algebras)

A root system in a real vector space E with a symmetric, positive def. bilinear form, is a nonempty finite set $\Phi \subset E \setminus \{0\}$ which spans E, which has $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ $\forall \alpha \in \Phi$, which has $r_\alpha(\Phi) = \Phi$ $\forall \alpha \in \Phi$, and which has $2(\beta,\alpha)/(\alpha,\alpha) \in \mathbb{Z}$ $\forall \alpha, \beta \in \Phi$.

* Call the integers $2(\beta, \alpha)/(\alpha, \alpha)$ the Cartan numbers of Φ
* An isomorphism between root systems is a bijection preserving Cartan numbers
* A root system is irreducible if it cannot be partitioned into disjoint nonempty orthogonal subsets

Key point: every root system has a unique decomposition as a disjoint union of irreducible subsystems (where a subset of Φ is viewed as a root system in the subspace of E that it spans)

The isomorphism classes of irreducible root systems belong to 7 families:

$A_n\ (n \geq 1)$, $B_n\ (n \geq 2)$, $C_n\ (n \geq 3)$, $D_n\ (n \geq 4)$

E_6, E_7, E_8, F_4, and G_2

Last time: we saw constructions in types ABCD

Choose a simple system / base Δ in a rootsystem Φ

↳ recall this is a basis for E such that we never need to mix signs of coefficients when expressing roots in terms of Δ.

order Δ as $\alpha_1\ \alpha_2\ \alpha_3 \ldots \alpha_n$.

Then the Cartan matrix of Φ is $\left[2(\alpha_i,\alpha_j)/(\alpha_j,\alpha_j)\right]_{1\leq i,j\leq n}$

The <u>Coxeter diagram</u> of Φ is the graph with vertex set Δ and with $\dfrac{4(\alpha_i,\alpha_j)(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)(\alpha_j,\alpha_j)} \in \mathbb{Z}_{\geq 0}$ edges between α_i and α_j for each $\alpha_i \neq \alpha_j$ in Δ.

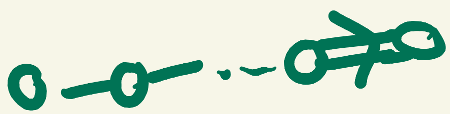
same

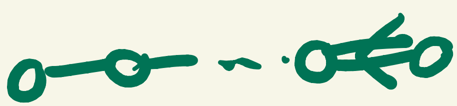
The <u>Dynkin diagram</u> of Φ is formed from the Coxeter diagram by orienting edges $\alpha = \beta$ and $\alpha \equiv \beta$ to be $\alpha \Rightarrow \beta$ and $\alpha \Rrightarrow \beta$ if $\|\alpha\| > \|\beta\|$. [Whenever a double or triple edge occurs, it is between roots of different lengths]

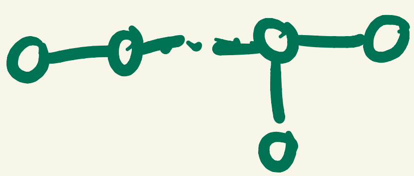
<u>Thm</u> The Cartan matrix and Dynkin diagram [up to arbitrary relabelling of indices / vertices] uniquely determine the isomorphism class of Φ. Also, Φ is irreducible iff the Dynkin diagram is connected.

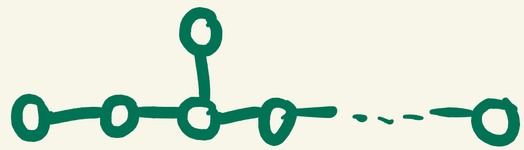
The graphs that occur as Dynkin diagrams of irreducible root systems are precisely

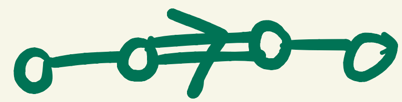
A_n : 

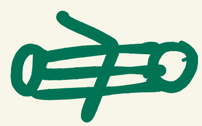
B_n : 

C_n : 

D_n : 

E_n :  $(n=6,7,8)$

F_n :  $(n=4)$

G_n :  $(n=2)$

[n vertices o in each picture]

Back to semisimple Lie algebras

Suppose L is a finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of char. zero. Assume L is semisimple [so L is direct sum of simple Lie algebras / has no solvable ideals]

"consists of all semisimple elems"

↳ "has no proper nonzero ideals and is nonabelian"

Then: for any maximal toral subalgebra $H \subseteq L$ there is a (unique) decomposition $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ for a certain subset $\Phi \subset H^*$,

where $L_\alpha \stackrel{\text{def}}{=} \{x \in L \mid [h, x] = \alpha(h)x \;\; \forall h \in H\}$.

The vector space H^* has a nondegenerate symmetric form dual to the restriction to H of the Killing form

$$\kappa(X,Y) \overset{\text{def}}{=} \operatorname{trace}(\operatorname{ad} X \operatorname{ad} Y) \text{ for } X, Y \in L.$$

The set Φ is a root system in H^*.

<u>Thm</u> L is simple iff Φ is irreducible.

Also, two semisimple Lie algebras are isomorphic if and only if their root systems are isomorphic.

one thing we haven't seen (for exceptional types):
For each irreducible root system there does exist a simple Lie algebra with that root system as its Φ.

Digression — Cartan subalgebras

Quick overview, no proofs

Def A <u>Cartan subalgebra</u> of a Lie algebra L is a nilpotent subalgebra $H \subseteq L$ with $H = N_L(H)$.

Here $N_L(H) \overset{\text{def}}{=} \{x \in L \mid [x,h] \in H \ \forall h \in H\}$

Thm If L is semisimple and defined over an alg. closed field $\mathbb{F}$ with $\operatorname{char}(\mathbb{F}) = 0$, then a subalgebra $H \subseteq L$ is a <u>maximal toral</u> subalgebra iff H is a <u>Cartan subalg.</u>

If $\operatorname{char}(\mathbb{F}) > 0$, then these are different, however.

Def A Borel subalgebra of a Lie algebra L is a maximal solvable subalgebra

Thm If B_1 and B_2 are two Borel subalgebras of a Lie algebra L, then there is an automorphism $f \in \mathrm{Aut}(L)$ with $f(B_1) = B_2$. Moreover, the same fact holds if B_1 and B_2 are two Cartan subalgebras.

Ex If $L = \mathfrak{sl}_n(\mathbb{F})$ then two Borel subalgebras are B_1 = upper Δ-matrices and B_2 = lower-Δ matrices. We have $f(B_1) = B_2$ for $f(x) = -x^T$.

Textbook proves stronger fact that $f \in \mathrm{Aut}(L)$ can be chosen in a subgroup $\mathcal{E}(L) \leq \mathrm{Aut}(L)$ generated by $\exp(\mathrm{ad}\, x)$ for $x \in L$ that are strongly ad-nilpotent in a certain sense.

Related important fact:

Cor In a general Lie algebra, all Cartan subalgebras H are isomorphic.

New topic today:

universal enveloping algebras.

The following constructions pertain to arbitrary Lie algebras over any field $\mathbb{F}$. The main idea is to construct from a Lie algebra L an associative unital algebra $\mathcal{U}(L)$ "as freely as possible" subject to the commutation relations of L. That is, we want to build the "most general possible" algebra $\mathcal{U}(L) \supseteq L$ that has

$$X \cdot Y - Y \cdot X = [X, Y] \quad \forall\, X, Y \in L.$$

An <u>associative unital algebra</u> is just a vector space A with an associative bilinear multiplication operation and a compatible unit $1 \in A$.

Def An <u>enveloping algebra</u> of a given Lie algebra L is a pair (A, ϕ) where A is an associative unital algebra and $\phi: L \to A$ is a linear map, such that

$$\phi([X,Y]) = \phi(X)\phi(Y) - \phi(Y)\phi(X) \quad \forall\, X, Y \in L.$$

Ex If $L \leq \mathfrak{gl}(V)$ for a vector space V then $\mathfrak{gl}(V)$ is an enveloping algebra wrt to obvious inclusion $\phi: L \hookrightarrow \mathfrak{gl}(V)$

A <u>morphism</u> of enveloping algebras $f: (A_1, \phi_1) \to (A_2, \phi_2)$ is an $\underbrace{\text{algebra homomorphism}}_{\substack{\text{a linear map sending units to units} \\ \text{commuting with multiplication}}}$ $f: A_1 \to A_2$ such that

$$\begin{tikzcd} A_1 \arrow[rr, "f"] & & A_2 \\ & L \arrow[ul, "\phi_1"] \arrow[ur, "\phi_2"'] & \end{tikzcd}$$

commutes

Def A universal enveloping algebra of L is an initial object in the category of enveloping algebras for L: that is, an enveloping algebra $(\mathcal{U}, i)$ such that if (A, ϕ) is any enveloping algebra for L then there is a unique morphism $(\mathcal{U}, i) \to (A, \phi)$

Prop If $(\mathcal{U}_1, i_1)$ and $(\mathcal{U}_2, i_2)$ are both universal enveloping algebras for L then there is a unique isomorphism $(\mathcal{U}_1, i_1) \xrightarrow{\sim} (\mathcal{U}_2, i_2)$

↳ morphism with a two-sided inverse, i.e. an algebra isomorphism commuting with relevant diagrams.

Pf By def., there are unique morphisms $f: (\mathcal{U}_1, i_1) \to (\mathcal{U}_2, i_2)$ and $g: (\mathcal{U}_2, i_2) \to (\mathcal{U}_1, i_1)$.
But the identity morphism is the only morphism $(\mathcal{U}_j, i_j) \to (\mathcal{U}_j, i_j)$. So $f \circ g = \mathrm{id}$, $g \circ f = \mathrm{id}$ $\square$

So there is at most one universal enveloping algebra of L (up to unique isomorphism). More involved:

<u>Thm</u> Any Lie algebra L has a universal enveloping algebra. [This is always infinite-dimensional if $L \neq 0$].

The proof requires a short digression on tensor algebras.

Let V be a finite-dimensional vector space over a field $\mathbb{F}$.

Define $T^0V = \mathbb{F}$

$T^1V = V$

$T^2V = V \otimes V$

$T^3V = V \otimes V \otimes V$

$\vdots$

$T^nV = V \otimes V \otimes \cdots \otimes V$ (n factors)

Let $T(V) = \bigoplus_{n \geq 0} T^nV$. This a vector space whose elements are finite linear combinations of tensors $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ for any $n \geq 0$, any $v_i \in V$

We make $T(V)$ into an associative unital algebra with unit $1 \in \mathbb{F} = T^0V \subset T(V)$ by setting

$$(v_1 \otimes \cdots \otimes v_k)(w_1 \otimes \cdots \otimes w_l) \overset{\text{def}}{=} v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_l$$

and extending by linearity, for $v_i, w_j \in V$. The resulting structure is the <u>tensor algebra</u> of V.

Some properties of the tensor algebra $T(V)$:

* associative
* graded as $T^m V \times T^n V \to T^{m+n} V$
* infinite-dim
* generated as an algebra by any basis of V

The tensor algebra of V is characterized by this universal property: for any associative unital algebra A and any linear map $\phi : V \to A$, there is a unique algebra morphism $\psi : T(V) \to A$ such that the diagram

$$\begin{tikzcd} V \arrow[rr, "\text{inclusion}"] \arrow[dr, "\phi"'] & & T(V) \arrow[dl, "\psi"] \\ & A & \end{tikzcd}$$

commutes.

Let I be the two-sided ideal in $T(V)$ generated by the set $\{ x \otimes y - y \otimes x \mid x, y \in V \}$

the intersection of all two-sided ideals containing these elements

The symmetric algebra of V is the quotient $S(V) \overset{\text{def}}{=} T(V)/I$

This is a commutative algebra with the same universal property as $T(V)$ but restricted to commutative algebras.

If $x_1, x_2, \dots, x_n$ is a basis for V then $S(V)$ is isomorphic to the polynomial algebra $\mathbb{F}[x_1, x_2, \dots, x_n]$ in n commuting variables.

The tensor algebra $T(V)$ is similarly isomorphic to the free associative algebra $\mathbb{F}\langle x_1, x_2, \dots, x_n \rangle$ of polynomials in n noncommuting variables.

Proof of existence of universal enveloping algebras (for Lie algebra L)

Let J be the two-sided ideal in $T(L)$ generated by the set $\{ \underbrace{x \otimes y}_{\in T^2 L} - \underbrace{y \otimes x}_{\in T^2 L} - \underbrace{[x,y]}_{\in T^1 L} \mid x, y \in L \}$.

Next we set $\mathcal{U}(L) \overset{\text{def}}{=} T(L)/J$. Also define $\pi : T(L) \to \mathcal{U}(L)$ to be the quotient map and define $i : L \to \mathcal{U}(L)$ to be the composition $L \hookrightarrow T(L) \xrightarrow{\pi} \mathcal{U}(L)$.

Since $J \subseteq \bigoplus_{n>0} T^n L$ the quotient $\mathcal{U}(L)$ is nonzero and contains $T^0 L = \mathbb{F}$

It is not yet clear whether or not i is injective (this will turn out to be true but is not part of any defs)

To show that $(\mathcal{U}(L), i)$ is a universal enveloping algebra:

suppose (A, j) is some enveloping algebra for L.

The universal property of $T(L)$ gives us a unique algebra homomorphism $\phi' : T(L) \to A$ such that the diagram

$$\begin{tikzcd} L \arrow[r, hook] \arrow[dr, "j"'] & T(L) \arrow[d, "\phi'"] \\ & A \end{tikzcd}$$

commutes.

But all elements $x \otimes y - y \otimes x - [x, y]$ for $x, y \in L$ are in $\mathrm{Ker}(\phi')$, since $\phi'(x \otimes y) = \phi'(x)\phi'(y)$ as ϕ' is algebra hom. Thus $J \leq \mathrm{ker}(\phi')$ so ϕ' descends to the desired unique morphism $(\mathcal{U}(L), i) \to (A, \phi)$. $\square$

<u>Ex</u>. Suppose L is abelian so that $[x,y]=0 \ \forall x,y \in L$.

Then $J = I$ and $\mathcal{U}(L) = S(L)$ is the symmetric algebra of L.

Next: algebra structure of $\mathcal{U}(L)$

and the Poincaré-Birkhoff-Witt theorem

describing a basis for $\mathcal{U}(L)$