MATH 5143 - Lecture #21

Representation theory of semisimple Lie algebras

Everywhere: L is semisimple Lie algebra over field $\mathbb{F}$

$\mathbb{F}$ is alg. closed and char. zero.

$H \subseteq L$ is a fixed Cartan subalgebra

$\Phi \subseteq H^*$ is the corresponding root system

$\Delta \subseteq \Phi$ is a simple system with elems $\alpha_1, \alpha_2, \ldots, \alpha_n$

$W \overset{\text{def}}{=} \langle r_\alpha \mid \alpha \in \Phi \rangle$ is Weyl group of Φ.

Goal: understand the finite-dimensional L-modules in particular those which are irreducible.

Suppose V is a finite-dim L module. Then H acts on V as commuting diagonalizable operators, so V can be decomposed into simultaneous eigenspaces for H.

Specifically, we can write $V = \bigoplus_{\lambda \in H^*} V_\lambda$

where $V_\lambda \overset{\text{def}}{=} \{ v \in V \mid h \cdot v = \lambda(h) v \ \forall h \in H \}$

If $V_\lambda \neq 0$ (which can only happen for finitely-many $\lambda \in H^*$) then we call V_λ a <u>weight space</u> and λ a <u>weight</u>

<u>Ex</u> If $V = L$, L acting by adjoint repn, then weight spaces are just the root spaces L_α and the weights are the roots $\alpha \in \Phi$

$\hookrightarrow$ along with H $\hookrightarrow$ along with 0

Ex If $L = sl_2(\mathbb{F}) = \left\langle x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$ and V is irreducible, then V looks like

$$V = V_{-m} \oplus V_{-m+2} \oplus \cdots \oplus V_{m-2} \oplus V_m$$

for some integer $m \geq 0$. Each V_i is a weight space for the weight $\lambda : h \mapsto i$. Everything is easy here because $H = \mathbb{F}\text{-span}\{h\}$.

Some pathologies: if $\dim V = \infty$ then the sum of the weight spaces $V_\lambda \leq V$ may be a proper subspace, though this sum of subspaces is always direct [HW exercise] However:

Lemma Let V be an arbitrary L-module. Then

(a) L_α maps V_λ into $V_{\lambda+\alpha}$ $\forall \lambda \in H^*$ and $\alpha \in \Phi$.

(b) $U \overset{\text{def}}{=} \sum_{\lambda \in H^*} V_\lambda$ is equal to $\bigoplus_{\lambda \in H^*} V_\lambda$ and is an L-submodule of V.

(c) If $\dim V < \infty$ then $U = V$.

Pf We will just check (a), as (b)(c) are exercises. Note for $x \in L_\alpha$, $v \in V_\lambda$, $h \in H$ that $h \cdot x \cdot v = x \cdot h \cdot v + [h, x] \cdot v = (\lambda(h) + \alpha(h))\, x \cdot v$ so L_α sends V_λ to $V_{\lambda+\alpha}$. $\square$

Standard cyclic modules

A **maximal vector** of weight $\lambda \in H^*$ in an L-module V is a nonzero vector $v^+ \in V$ with $\begin{cases} x v^+ = 0 \ \forall \alpha \in \Delta, x \in L_\alpha \\ h v^+ = \lambda(h) v^+ \ \forall h \in H \end{cases}$

[This depends implicitly on choice of simple roots Δ.
If $\dim V = \infty$ then it could happen that there are no such vectors]

But if $\dim V < \infty$ then the Borel subalgebra $B = H \oplus \bigoplus_{\alpha \in \Phi^+} L_\alpha$ is solvable and so has a common eigenvector in V (by Lie's thm) and this eigenvector provides a maximal vector (because it is killed by all L_α for $\alpha \in \Phi^+$). $\boxed{\text{Idea:}}$ first study L-modules generated by a maximal vector.

Note that any L-module structure on V corresponds to a map $L \to \mathfrak{gl}(V)$ which is an algebra, and so extends uniquely to an associative algebra module structure on V relative to $\mathcal{U}(L)$.

If $V = \mathcal{U}(L) \cdot v^+$ for a maximal vector v^+ of weight λ, then we say V is <u>standard cyclic</u> of weight λ, and we call v^+ the <u>highest weight vector</u> of V.

Fix $x_\alpha \in L_\alpha$, $y_\alpha \in L_{-\alpha}$ with $[x_\alpha, y_\alpha] = h_\alpha$ for each $\alpha \in \Phi^+$.

Write $\lambda > \mu$ for $\lambda, \mu \in H^*$ if $\lambda - \mu$ is a sum of positive roots

<u>Thm</u> Let V be a standard cyclic L-module with highest weight vector $v^+ \in V_\lambda$. Write $\Phi^+ = \{\beta_1, \beta_2, \dots, \beta_m\}$ and $y_i \overset{\text{def}}{=} y_{\beta_i}$. Then:

(a) V is spanned by the vectors $y_{i_1} y_{i_2} \cdots y_{i_k} v^+$ as $(i_1, i_2, \dots, i_k)$ ranges over all weakly increasing sequences $1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq m$. Also V is the direct sum of its weight spaces

(b) All weights μ for V have the form

$$\mu = \lambda - \sum_{i=1}^{n} k_i \alpha_i \quad (\text{where } k_i \in \mathbb{Z}_{\geq 0})$$

and therefore $\mu < \lambda$.

(c) For each $\mu \in H^*$, $\dim V_\mu < \infty$ and $\dim V_\lambda = 1$

(d) Each submodule of V is a direct sum of weight spaces

(e) V is an indecomparable L-module with a unique maximal proper submodule whose quotient is irreducible.

(f) every nonzero homomorphic image of V is also standard cyclic of weight λ.

Pf Let $N^- = \bigoplus_{\alpha \in \Phi^-} L_\alpha$ and $B = H \oplus \bigoplus_{\alpha \in \Phi^+} L_\alpha$ so $L = N^- \oplus B$.

PBW thm implies that $\mathcal{U}(L)v^+ = \mathcal{U}(N^-)\mathcal{U}(B)v^+ = \mathcal{U}(N^-)\mathbb{F}v^+$ since v^+ is common eigenvector for B. Part (a) follows from PBW thm for N^-.

Our lemma above implies that $y_{i_1} y_{i_2} \cdots y_{i_k} v^+$ has weight

$$(*) \quad \mu = \lambda - \beta_{i_1} - \beta_{i_2} - \cdots - \beta_{i_k}$$

so part (b) also follows. There are only finitely many vectors in (a) that can give rise a given weight μ via $(*)$ so $\dim V_\mu < \infty$, and the only such weight vector of weight λ is v^+ so $\dim V_\lambda = 1$.

For part (d), let W be a submodule of V and write $w \in W$ as a sum of vectors $v_i \in V_{\mu_i}$ for distinct weights μ_i.

We want to show that each v_i is in W.

Suppose otherwise and choose $w = v_1 + \dots + v_n$ with n minimal where none of $v_1, v_2, \dots, v_n$ are in W. (Then $n>1$)

Find $h \in H$ with $\mu_1(h) \neq \mu_2(h)$. Then $h \cdot w = \sum_i \mu_i(h) v_i \in W$ so $(h - \mu_1(h)) w \in W$ but $(h-\mu_1(h))w$ has the form

$$(\mu_2(h) - \mu_1(h)) v_2 + \dots + (\mu_n(h) - \mu_1(h)) v_n \neq 0$$

contradicting minimality of n. Hence each $v_i \in W$ and (d) holds.

We conclude from (c) and (d) that each proper submodule of V is in sum of weight spaces other than V_λ, so the sum W of all proper submodules is proper, so the quotient V/W must be irreducible.

This proves (e), and (f) holds by definition. $\square$

<u>Cor</u> If V is as in thm and V is irreducible then v^+ is the unique maximal weight vector up to rescaling.

<u>Pf</u> If there were another such vector of weight λ' then thm implies that $\lambda < \lambda'$ and $\lambda' < \lambda$ so $\lambda = \lambda'$. $\square$