MATH 5143 – Lecture #24

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Rad $L = 0 \iff$ no solvable ideals $\iff$ no abelian ideals
$\iff L = \bigoplus$ (simple Lie alg.)

Setup: L is a semisimple Lie algebra defined over an algebraically closed, char. zero field $\mathbb{F}$. Assume $\dim L < \infty$, let $H \subseteq L$ be a Cartan subalgebra, write $\Phi \subset H^*$ for corresponding root system so that $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$, choose a simple system $\Delta \subseteq \Phi$

If V is any L-module (meaning $[x,y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v$) then a weight space in V is any nonzero subspace of form $V_\lambda = \{v \in V \mid h \cdot v = \lambda(h) v \ \forall h \in H\}$ for $\lambda \in H^*$. Call λ the weight

Last time Fact Any finite-dim L-module is direct sum of its weight spaces

A standard cyclic L-module [of weight $\lambda \in H^*$] is an L-module V generated by a maximal vector v^+ (to be called the highest weight vector) with
$$\begin{cases} x \cdot v^+ = 0 & \forall \alpha \in \Delta \ \forall x \in L_\alpha \\ h \cdot v^+ = \lambda(h) v^+ & \forall h \in H \end{cases}$$

Fact Every finite-dim L-module is standard cyclic
Every standard cyclic L-module is direct sum of its weight spaces

Thm Suppose V is a standard cyclic L-module of weight λ
then all weights μ for V have $\dim V_\mu < \infty$ and $\underbrace{\mu < \lambda}_{}$ and $\dim V_\lambda = 1$

means $\lambda - \mu \in \mathbb{Z}_{\geq 0}$-span$[\Delta]$

also V has a unique maximal proper submodule, and unique irreducible quotient, and all homomorphic images of V are also standard cyclic of weight λ

<u>Thm</u> For each $\lambda \in H^*$ there exists a unique isomorphism class of <u>irreducible</u> standard cyclic L-modules $V(\lambda)$ of weight λ. One explicit construction:

Define $\boxed{Z(\lambda) = \mathcal{U}(L) \otimes_{\mathcal{U}(B)} D_\lambda}$ infinite-dim, not nec. irreducible, but it is standard cyclic of wt λ

where $B = H \oplus \bigoplus_{\alpha \in \Phi^+} L_\alpha$ and $D_\lambda = \mathbb{F}\text{-span}\{v^+\}$

with B acting on D_λ such that $\begin{cases} hv^+ = \lambda(h)v^+ & \forall h \in H \\ xv^+ = 0 & \forall x \in L_\alpha, \alpha \in \Phi^+ \end{cases}$

In this tensor product, we have $xb \otimes v^+ = x \otimes bv^+ \quad \forall b \in \mathcal{U}(B)$

Finally define $V(\lambda) \overset{\text{def}}{=} \big(\text{unique irreducible quotient of } Z(\lambda)\big)$

__Fact__ If V is any finite dim. irreducible L-module then $V \cong V(\lambda)$ for some $\lambda \in H^*$.

Call $\lambda \in H^*$ __dominant__ if $\lambda(h_\alpha) > 0 \ \forall \alpha \in \Delta$

__integral__ if $\lambda(h_\alpha) \in \mathbb{Z} \ \forall \alpha \in \Delta$

__dominant integral__ if $\lambda(h_\alpha) \in \mathbb{Z}_{\geq 0} \ \forall \alpha \in \Delta$

Let Λ be set of integral $\lambda \in H^*$, let $\Lambda^+ \subset \Lambda$ be subset of dominant integral $\lambda \in \Lambda$

Here: $h_\alpha \stackrel{\text{def}}{=} \dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$ where $\kappa : L \times L \to \mathbb{F}$ is killing form
$\kappa(x,y) = \text{trace}(\text{ad}\,x \ \text{ad}\,y)$

and $t_\alpha \in H$ is unique elem with $\kappa(t_\alpha, h) = \alpha(h) \ \forall h \in H$

Thm Let $\lambda \in H^*$. Then the irreducible L-module $V(\lambda)$ is finite-dimensional if and only if $\lambda \in \Lambda^+$ is dominant integral. Moreover, in this case the Weyl group W of Φ permutes the set of weights in $V(\lambda)$ and weights in same W-orbit all have weight spaces of same dimension.

Define $m_\lambda(\mu) \overset{\text{def}}{=} \dim V(\lambda)_\mu$ for $\mu \in H^*$.

There are various formulas/recurrences to explicitly compute these numbers, e.g. Freudenthal's formula (see Lecture #23)

To make it easier to compute and manipulate the multiplicity function m_λ, we encode it as a formal character:

Let $\mathbb{Z}[\Lambda]$ be $\mathbb{Z}$-span of the symbols e^λ for $\lambda \in \Lambda$ viewed as a ring with $e^\lambda e^\mu \overset{\text{def}}{=} e^{\lambda+\mu}$ and $1 = e^0$.

The formal character of any fin. dim. L-module V is

$$\mathrm{ch}_V \overset{\text{def}}{=} \underbrace{\sum_{\mu \in H^*} m_\lambda(\mu) e^\mu}_{\text{finite sum}} \in \mathbb{Z}[\Lambda]$$

W acts linearly on $\mathbb{Z}[\Lambda]$ by $w \cdot e^\lambda = e^{w\lambda}$ and this action fixes ch_V for any V.

Prop Any $f \in \mathbb{Z}[\Lambda]$ fixed by all $w \in W$ has unique expansion as linear comb. of formal characters $\mathrm{ch}_\lambda \overset{\text{def}}{=} \mathrm{ch}_{V(\lambda)}$

Prop If V and W are two finite-dim L-modules then $ch_{V\otimes W} = ch_V \, ch_W$

Today: Harish-Chandra's theorem and applications

Some technical proofs will just be outlined

Let Z denote the center of the algebra $\mathfrak{U}(L)$:

$$Z \overset{\text{def}}{=} \{ x \in \mathfrak{U}(L) \mid xy = yx \ \forall y \in L \}$$

This is a commutative subalgebra, and each of L-module is also a $\mathfrak{U}(L)$-module and, by restriction, a Z-module.

Consider the standard cyclic L-module

$$Z(\lambda) = \mathfrak{U}(L) \otimes_{\mathfrak{U}(B)} D_\lambda$$

for some $\lambda \in H^*$, now viewed as a $\mathcal{Z}$-module.

If v^+ is a maximal vector in $Z(\lambda)$ and $z \in \mathcal{Z}$

then
$$\begin{cases} h \cdot z \cdot v^+ = z \cdot h \cdot v^+ = \lambda(h) z \cdot v^+ & \forall h \in H \\ x \cdot z \cdot v^+ = z \cdot x \cdot v^+ = 0 & \forall \begin{smallmatrix} \alpha \in \Delta \\ x \in L_\alpha \end{smallmatrix} \end{cases}$$

Thus $z \cdot v^+$ is also a maximal vector of weight λ.

Therefore $z \cdot v^+$ is a scalar multiple of v^+.

Define $\chi_\lambda : \mathcal{Z} \to \mathbb{F}$ to be map with $z \cdot v^+ = \chi_\lambda(z) v^+$ $\forall z \in \mathcal{Z}$

Does not depend on choice of v^+, as all maximal vectors in $Z(\lambda)$ are scalar multiples of each other.

<u>Fact</u> χ_λ is an algebra homomorphism

<u>Pf</u> $\chi_\lambda(z_1 z_2)v^+ = z_1 z_2 \cdot v^+ = z_1 \cdot (z_2 \cdot v^+)$

$= \chi_\lambda(z_2)\, z_1 \cdot v^+ = \chi_\lambda(z_1)\chi(z_2)v^+.$ $\square$

Call $\chi_\lambda : \left(Z = \text{center of } \mathcal{U}(L) \right) \to \mathbb{F}$

the <u>(central) character</u> of $\lambda \in H^*$ [or of $Z(L)$]

These central character χ_λ may coincide for different λ's, and Harish-Chandra's thm will tell us precisely when this happens.

Fact If $z \in \mathfrak{Z}$ and $u \in Z(\lambda)$ is any vector then

$$z \cdot u = \chi_\lambda(z) u.$$

Pf Since v^+ generates $Z(\lambda)$ and z commutes with all elements of L, the result follows. $\square$

Cor The action of $z \in \mathfrak{Z}$ on any submodule of homomorphic image of $Z(\lambda)$ is by the scalar $\chi_\lambda(z)$

Def Two elements $\lambda, \mu \in H^*$ are <u>linked</u> (by $w \in W$)

if $\lambda + \delta = w \cdot (\mu + \delta)$ where $\delta \overset{\text{def}}{=} \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$

In this situation we write $\mu \sim \lambda$.

Given $\alpha \in \Phi^+$ choose $0 \neq x_\alpha \in L_\alpha$. Then there exists a unique $y_\alpha \in L_{-\alpha}$ such that if $h_\alpha \overset{\text{def}}{=} [x_\alpha, y_\alpha]$ then $\langle x_\alpha, y_\alpha, h_\alpha \rangle \cong sl_2(\mathbb{F})$

via $x_\alpha \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $y_\alpha \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$h_\alpha \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Prop Let $\lambda \in \Lambda$, $\alpha \in \Delta$, $m = \langle \lambda, \alpha \rangle \in \mathbb{Z}$.

If $m \geq 0$ then $y_\alpha^{m+1} \otimes_{\mathcal{U}(B)} v^+ \in Z(\lambda)$ is a maximal vector of weight $\lambda - (m+1)\alpha$. [Here, $v^+ \in D_\lambda$ so $1 \otimes_{\mathcal{U}(B)} v^+$ generates $Z(\lambda)$]

Pf Formulas last time tell us that:

For $\alpha \neq \beta$ in Δ: $[x_\beta, y_\alpha^{m+1}] = 0 \Rightarrow x_\beta \cdot y_\alpha^{m+1} \otimes v^+ = y_\alpha^{m+1} \otimes \underbrace{x_\beta v^+}_{=0} = 0$

For any $\alpha, \beta \in \Delta$: $[h_\beta, y_\alpha^{m+1}] = -(m+1)\,\alpha(h_\beta)\, y_\alpha^{m+1}$

$$\Rightarrow h_\beta \cdot y_\alpha^{m+1} \otimes v^+ = -(m+1)\,\alpha(h_\beta)\, y_\alpha^{m+1} \otimes v^+ + y_\alpha^{m+1} \otimes \underbrace{h_\beta v^+}_{= \lambda(h_\beta)}$$

$$= (\lambda - (m+1)\alpha)(h_\beta)\; y_\alpha^{m+1} \otimes v^+ \qquad \square$$

If λ and μ are linked by $r_\alpha \in W$ then $\chi_\lambda = \chi_\mu$

Cor If $\boxed{\lambda \in \Lambda}$, $\alpha \in \Delta$, $\mu = r_\alpha \cdot (\lambda + \delta) - \delta$ where $r_\alpha \in W$
$x \mapsto x - \langle x, \alpha \rangle \alpha$

then $\chi_\lambda = \chi_\mu$ $\quad (\delta = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta)$

Pf r_α sends $\alpha \mapsto -\alpha$ and permutes $\Phi^+ \setminus \{\alpha\}$, so $r_\alpha \delta - \delta = -\alpha$ and

$\mu = r_\alpha \lambda - \alpha = \lambda - (\langle \lambda, \alpha \rangle + 1)\alpha$. we always have $\langle \lambda, \alpha \rangle \in \mathbb{Z}$.

If $\langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0}$ then previous prop shows that $Z(\lambda)$ has maximal vector of weight μ.

As $z \in \mathcal{Z}$ acts on this vector by the scalar $\underbrace{\chi_\mu(z)}_{\text{as the maximal vector of wt } \mu \text{ generates a homomorphic image of } Z(\mu)}$ and $\underbrace{\text{also } \chi_\lambda(z)}_{\text{by earlier observations}}$, we must have $\chi_\lambda = \chi_\mu$.

If $\langle \lambda, \alpha \rangle < 0$ then

$$\langle \mu, \alpha \rangle = \langle \lambda, \alpha \rangle - 2(\langle \lambda, \alpha \rangle + 1) = -\langle \lambda, \alpha \rangle - 2$$

is ≥ 0 so we can apply proposition with μ in place of λ to deduce the same conclusion. $\square$

Because $W = \langle r_\alpha \mid \alpha \in \Delta \rangle$ we can conclude:

<u>Cor</u> (Easy direction of Harish-Chandra's thm)

If $\lambda \sim \mu$ where $\lambda \in \Lambda$ then $\chi_\lambda = \chi_\mu$.

<u>Thm</u> (Harish-Chandra's thm) Let $\lambda, \mu \in H^*$.

Then $\chi_\lambda = \chi_\mu$ if and only if $\underbrace{\lambda \sim \mu}$

means $\lambda + \delta$ and $\mu + \delta$ are in the same W-orbit

Outline of proof of Harish-Chandra thm

First part: already know that $\lambda \sim \mu \Rightarrow \chi_\lambda = \chi_\mu$ when $\lambda \in \Lambda$

we want to extend this to a statement allowing any $\lambda \in H^*$

Construct PBW bases of $\mathfrak{U}(L)$ and $\mathfrak{U}(H)$ from the basis $\{h_\alpha \mid \alpha \in \Delta\} \sqcup \{x_\alpha, y_\alpha \mid \alpha \in \Phi^+\}$ for L, under any order putting all y_α's first, then the h_α's, then the x_α's.

Then we can define a linear map $\xi : \mathfrak{U}(L) \to \mathfrak{U}(H)$ sending each PBW basis elem in $\mathfrak{U}(H)$ to itself, every other PBW basis elem to 0.

Since $\prod_{\alpha \in \Phi^+} y_\alpha^{i_\alpha} \prod_{\alpha \in \Delta} h_\alpha^{k_\alpha} \prod_{\alpha \in \Phi^+} x_\alpha^{j_\alpha}$ will either $\begin{cases} \text{kill } v^+ \in Z(\lambda) \text{ if any } j_\alpha > 0 \\ \text{send } v^+ \text{ to lower weight space if all } j_\alpha = 0 \text{ and any } i_\alpha > 0 \end{cases}$

it follows that $\boxed{\chi_\lambda(z) = \lambda(\xi(z))}$ $\forall z \in \mathfrak{Z}$

Now define another Lie algebra homomorphism $\eta: H \to \mathcal{U}(H)$ with $\eta(h_\alpha) = h_\alpha - 1 \;\forall \alpha \in \Delta$. This extends to an algebra automorphism $\eta: \mathcal{U}(H) \to \mathcal{U}(H)$. Define

$$\psi: \mathcal{Z} \xrightarrow{\xi} \mathcal{U}(H) \xrightarrow{\eta} \mathcal{U}(H) \quad (\psi = \eta \circ \xi)$$

We can write $\delta = \sum_{\alpha \in \Delta} \lambda_\alpha$ as sum over fundamental weights λ_α which have $\lambda_\alpha(h_\beta) = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$ for $\alpha, \beta \in \Delta$. Then we have

$$(\lambda+\delta)(h_\alpha - 1) = \underbrace{(\lambda+\delta)(h_\alpha)}_{\lambda(h_\alpha)+1} - \underbrace{(\lambda+\delta)(1)}_{=1} = \lambda(h_\alpha)$$

so $(\lambda+\delta)(\psi(z)) = \lambda(\xi(z)) \;\forall z \in \mathcal{Z}, \lambda \in H^*$

$$\Rightarrow \boxed{(\lambda+\delta)(\psi(z)) = \chi_\lambda(z) \;\forall z \in \mathcal{Z}, \lambda \in H^*}$$

Now check that $\psi(z)$ is W-invariant

(using the easy case of theorem and properties of W-orbits in Λ) and use this to conclude that

if $\lambda \sim \mu$ then $(\lambda+\delta)(\psi(z)) = (\mu+\delta)(\psi(z))$

and hence that $\chi_\lambda = \chi_\mu$, (for any $\lambda, \mu \in H^*$).

The other half of the theorem remains:

if $\chi_\lambda = \chi_\mu$ then need to show that $\lambda \sim \mu$.

This requires a more involved argument $\rightsquigarrow$ see §23.3 of textbook. $\square$

Applications of Harish-Chandra thm

We want to introduce formal characters for $Z(\lambda)$ and similar modules

Let $\mathfrak{X}$ be the vector space of all formal $\mathbb{Z}$-linear combinations

$$\sum_{\lambda \in H^*} c_\lambda e^\lambda \quad (c_\lambda \in \mathbb{Z},\ e^\lambda \text{ is a symbol})$$

which are finitely supported in the sense that

there are finitely many $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k \in H^*$ such that

$$c_\lambda \neq 0 \Rightarrow \lambda \leq \lambda_i \text{ for some } i$$

where $\lambda \leq \mu$ means $\mu - \lambda \in \mathbb{Z}_{\geq 0}\text{-span}\{\alpha \in \Delta\}$

Then the formal character $\boxed{ch_{Z(\lambda)} \overset{def}{=} \sum_{\mu \in H^*} \dim Z(\lambda)_\mu e^\mu}$

belongs to $\mathfrak{X}$.

__Fact__ $\mathcal{X}$ is closed under usual multiplication extending ring structure on $\mathbb{Z}[\Lambda]$. (Since $\lambda_i \leq \mu_i \Rightarrow \lambda_1 + \lambda_2 + \dots \leq \mu_1 + \mu_2 + \dots$)

We now have a well-defined notion of formal character

$$\mathrm{Ch}_V \overset{\text{def}}{=} \sum_{\mu \in H^*} \dim V_\mu e^\mu \in \mathcal{X}$$

for any standard cyclic L-module V.

Let $p(\lambda)$ for $\lambda \in H^*$ be # of functions $k : \Phi^+ \to \mathbb{Z}_{\geq 0}$ such that $\boxed{\lambda + \sum_{\alpha \in \Phi^+} k(\alpha)\alpha = 0}$ clearly $p(\lambda) = 0$ unless $(-\lambda) \in \mathbb{Z}_{\geq 0}\text{-span}\{\alpha \in \Delta\}$

Call p the __Kostant (partition) function__ and identify $p \leftrightarrow \sum_{\lambda \in H^*} p(\lambda) e^\lambda \in \mathcal{X}$

Also let $q = \prod_{\substack{\alpha \in \Phi^+ \\ \text{finite}}} (\underbrace{e^{\alpha/2}}_{} - \underbrace{e^{-\alpha/2}}_{})$ call this the <u>Weyl function</u>

symbols since $\pm\alpha/2 \in H^*$

Finally set $f_\alpha = e^0 + e^{-\alpha} + e^{-2\alpha} + \cdots \in \mathcal{X}$ for $\alpha \in \Phi^+$.

Lemma A (a) $p = \prod_{\alpha \in \Phi^+} f_\alpha$ (b) $(e^0 - e^{-\alpha}) f_\alpha = e^0$

(c) $q = e^\delta \prod_{\alpha \in \Phi^+} (e^0 - e^{-\alpha})$ where $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$

<u>Pf</u> (a) holds by defn, (b) is basic algebra (c) is clear. □

Lemma B For any $w \in W$ it holds that $wq = \mathrm{sgn}(w)q$

W acts linearly on $\mathcal{X}$ by $w \cdot e^{\lambda} = e^{w\lambda}$

Pf Suffices to show $r_\alpha q = -q$ for any $\alpha \in \Delta$

Easy enough: $r_\alpha q = r_\alpha\left(e^{\alpha/2} - e^{-\alpha/2}\right) r_\alpha \Big(\underbrace{\prod_{\beta \in \Phi^+ \setminus \{\alpha\}} \left(e^{\beta/2} - e^{-\beta/2}\right)}_{\text{fixed by } r_\alpha} \Big)$

$r_\alpha : \pm\frac{\alpha}{2} \mapsto \mp\frac{\alpha}{2}$

$= -q.$ $\square$

Lemma C $qpe^{-\delta} = e^0 = 1$

Pf $qpe^{-\delta} = \prod_{\alpha \in \Phi^+} (e^0 - e^{-\alpha}) \cdot e^{\delta} \cdot p \cdot e^{-\delta} = \prod_{\alpha \in \Phi^+} (e^0 - \alpha^{-\alpha}) p$

$= \prod_{\alpha \in \Phi^+} \left((e^0 - e^{-\alpha}) f_\alpha \right) = \prod_{\alpha \in \Phi^+} e^0 = e^0 = 1$ (using Lemma A)

$\boxed{\text{Lemma D}}$ $\quad ch_{Z(\lambda)} = \sum_{\mu \in H^*} p(\mu-\lambda)e^{\mu} = e^{\lambda} \underbrace{p}_{\sum_{\mu \in H^*} p(\mu)e^{\mu}}$

Pf Straightforward from properties of $Z(\lambda)$ □
(see Ex. 20.5 in textbook)

$\boxed{\text{Lemma E}}$ $\quad q\, ch_{Z(\lambda)} = e^{\lambda+\delta}$

Pf $qpe^{-\delta} = e^{0} = 1$ and $ch_{Z(\lambda)} = e^{\lambda} p$

so $\quad q\, ch_{Z(\lambda)} = e^{\lambda+\delta} qpe^{-\delta} \underset{\substack{\uparrow \\ \text{Lemma C}}}{=} e^{\lambda+\delta}$ □

Want to express $ch_\lambda = ch_{V(\lambda)}$ as linear comb. of $ch_{Z(\mu)}$'s.

Define M_λ (for $\lambda \in H^*$) to be the family of L-modules V

such that (1) V is direct sum of its weight spaces

(2) $\mathfrak{Z}$-action on V is by scalar $\chi_\lambda(z)$

(3) $ch_V \in \mathcal{X}$

M_λ is closed under taking submodules, homomorphic images, direct sums, contains each standard cyclic module.

Cor (of Harish-Chandra thm) $M_\lambda = M_\mu$ iff $\lambda \sim \mu$

Lemma Suppose $0 \neq V \in \mathcal{M}_1$. Then V has a maximal vector.

Pf Since $\mathrm{ch}_V \in \mathfrak{X}$, for each weight μ of V, and each $\alpha \in \Phi^+$ there is a maximal $k \in \mathbb{Z}_{\geq 0}$ with $\mu + k\alpha$ still a weight. So we can find a weight μ for V such that $\mu + \alpha$ is not a weight $\forall \alpha \in \Phi^+$, and then any nonzero vector in the corresponding weight space is maximal $\square$

For $\lambda \in H^*$, let $\Theta(\lambda) = \{\mu \in H^* \mid \mu < \lambda \text{ and } \mu \sim \lambda\}$.

<u>Prop</u> Let $\lambda \in H^*$. (a) $Z(\lambda)$ has a composition series.

(b) Each composition factor of $Z(\lambda)$ is $\cong V(\mu)$ for some $\mu \in \Theta(\lambda)$

(c) $V(\lambda)$ occurs as exactly one composition factor.

<u>Pf</u> (a) Nothing to prove if $Z(\lambda)$ is irreducible. (then $Z(\lambda) = V(\lambda)$)

Otherwise $Z(\lambda)$ has a proper nonzero submodule $V \in M_\lambda$.

Since $\dim Z(\lambda)_\lambda = 1$, λ is not a weight of V. So by

lemma, V has maximal vector, of some weight $\mu \lneq \lambda$.

V contains homomorphic image W of $Z(\mu)$, so $\chi_\lambda = \chi_\mu \Rightarrow \lambda \sim \mu$

$\Rightarrow \mu \in \Theta(\lambda)$. Continue inductively, repeating same argument applied to W and $Z(\lambda)/W$

(b) Each comp. factor is in $\mathcal{M}_\lambda$ so has a maximal vector and is irreducible, so must be standard cyclic, hence $\cong V(\mu)$ for some $\mu \in \Theta(\lambda)$

(c) Clear since $\dim Z(\lambda)_\lambda = 1$. $\square$

Cor Let $\lambda \in H^*$. Then $ch_{V(\lambda)} = \sum_{\mu \in \Theta(\lambda) = \{\nu \in H^* \mid \nu \leq \lambda, \nu \sim \lambda\}} c_\mu \, ch_{Z(\mu)}$

for some coeffs $c_\mu \in \mathbb{Z}$ with $c_\lambda = 1$.

Pf Prop. says we can write $ch_{Z(\lambda)} = ch_{V(\lambda)} + \sum_{\mu \in \Theta(\lambda)} d_\mu \, ch_{V(\mu)}$ where $d_\mu \in \mathbb{Z}_{\geq 0}$. Thus $ch_{V(\lambda)} = ch_{Z(\lambda)} - \sum_{\mu \in \Theta(\lambda)} d_\mu \, ch_{V(\mu)}$ and expanding the RHS recursively gives desired formula. $\square$

Thm (Kostant's formula) Let $\lambda \in \Lambda^+$ then

$$m_\lambda(\mu) = \sum_{\underbrace{w \in W}_{\text{potentially many terms}}} \operatorname{sgn}(w)\, p(\mu+\delta - w(\lambda+\delta))$$

Pf $ch_\lambda = \sum_{\mu \in \Theta(\lambda)} c_\mu \, ch_{Z(\mu)}$ with $c_\lambda = 1$. Lemmas E and B

tell us that
$$\begin{cases} q\, ch_\lambda = \sum_{\mu \in \Theta(\lambda)} c_\mu e^{\mu+\delta} \quad \text{and} \\ w(q\, ch_\lambda) = w(q)\, w(ch_\lambda) = \operatorname{sgn}(w)\, q\, ch_\lambda \quad \forall w \in W. \end{cases}$$

But also $w\left(\sum_{\mu \in \Theta(\lambda)} c_\mu e^{\mu+\delta}\right) = \sum_{\mu \in \Theta(\lambda)} c_\mu e^{w(\mu+\delta)}$ since $w \in W$

permutes $\Theta(\lambda)$ while $c_\lambda = 1$, deduce that $c_\mu = \operatorname{sgn}(w)$ if $\underbrace{w^{-1}(\mu+\delta)}_{=\lambda+\delta}$

So $\boxed{q\, ch_\lambda = \sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\delta)}}$

By Lemma C, $ch_\lambda = q p e^{-\delta} ch_\lambda$

$$= p e^{-\delta} \left(\sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\delta)} \right)$$

$$= p \sum_{w \in W} \operatorname{sgn}(w)\, e^{w(\lambda+\delta)-\delta}$$

$$= \sum_{w \in W} \operatorname{sgn}(w)\, p\, e^{w(\lambda+\delta)-\delta} \quad \square$$

<u>Cor</u> $q = \sum_{w \in W} \operatorname{sgn}(w) e^{w\delta}$

<u>Pf</u> Take $\lambda = 0$. $\square$

$\boxed{\text{Next time}}$: Weyl character formula