Instructions: Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due on **Tuesday, February 20**.

Throughout, let $\mathbf{k}$ be an algebraically closed field and let A be an algebra over $\mathbf{k}$.

1. Show that any finite-dimensional representation of A has an irreducible subrepresentation.

   (Remember that irreducible representations are required to be nonzero.)

2. Show that the regular representation of $\mathbf{k}[x]$ has no irreducible subrepresentations.

3. Let $Z(A) = \{z \in A : az = za \text{ for all } a \in A\}$. This is called the *center* of A, and it is a subalgebra.

   Suppose (ρ, V) is an irreducible finite-dimensional representation of A. Show that there exists an algebra morphism $\chi : Z(A) \to \mathbf{k}$ such that $\rho(z)(v) = \chi(z)v$ for all $z \in Z(A)$ and $v \in V$.

   The map χ is called the *central character* of (ρ, V).

4. Suppose (ρ, V) is an indecomposable finite-dimensional representation of A.

   Show that if $z \in Z(A)$ then $\rho(z)$ has exactly one eigenvalue $\chi(z) \in \mathbf{k}$ as a linear operator $V \to V$, and that the corresponding map $\chi : Z(A) \to \mathbf{k}$ is again an algebra morphism.

   To prove this, you may wish to use the following result from linear algebra. Suppose $L : V \to V$ is a linear operator. The *generalized eigenspace* of $\lambda \in \mathbf{k}$ is the subspace

   $$V_\lambda = \{v \in V : (L - \lambda)^m v = 0 \text{ for some } m \geq 1\}.$$

   An element $\lambda \in \mathbf{k}$ is called a *generalized eigenvalue* of L if $V_\lambda \neq 0$. Finally, it holds that $V = \bigoplus_\lambda V_\lambda$ where the internal direct sum is over the finite set of generalized eigenvalues of L.

5. Define $\rho_b : A \to A$ for $b \in A$ to be map with $\rho_b(a) = ab$ for all $a \in A$.

   Check that $\{\rho_b : b \in A\}$ is a subalgebra of $\mathrm{End}(A)$ isomorphic to A^{op}. Then show that each morphism from the regular representation of A to itself has the form ρ_b for a unique $b \in A$.

   This shows that we have $\mathrm{End}_A(A) \cong A^{\mathsf{op}}$ as algebras, where if V is a left A-module then $\mathrm{End}_A(V)$ denotes the algebra of morphisms of left A-modules $V \to V$.

6. Consider the algebra $\mathbf{k}(x)$ of rational functions in one variable over $\mathbf{k}$.

   Show that the dimension of $\mathbf{k}(x)$ as a $\mathbf{k}$-vector space is at least the cardinality of $\mathbf{k}$.

   You can do this by checking that the functions $\frac{1}{x-\lambda}$ for $\lambda \in \mathbf{k}$ are linearly independent.

7. Suppose (ρ, V) is an irreducible representation of A.

   Write $\mathrm{End}(\rho, V)$ for the algebra of morphisms $(\rho, V) \to (\rho, V)$.

   (a) Show that if $\phi \in \mathrm{End}(\rho, V)$ is not a scalar map $V \to V$, then there exists an injective algebra morphism $\mathbf{k}(x) \to \mathrm{End}(\rho, V)$ with $x \mapsto \phi$.

   (b) Show that if $0 \neq v \in V$ is fixed, then the linear map $\mathrm{End}(\rho, V) \to V$ given by $\phi \mapsto \phi(v)$ is injective.

   Conclude that $\dim \mathrm{End}(\rho, V) \leq \dim V$ so if $\dim V < |\mathbf{k}|$ then every $\phi \in \mathrm{End}(\rho, V)$ is a scalar.

   This applies when $\mathbf{k} = \mathbb{C}$ and V has a countable basis; this result is called Dixmier's lemma.

8. Fix a positive integer N. Suppose $A = \mathbf{k}[x_1, x_2, \ldots, x_n]$ and $I \neq A$ is a proper ideal containing all homogeneous polynomials of degree at least N. Show that the quotient A/I of the regular representation is an indecomposable representation of A.

9. Let (ρ, V) be a representation of A. A nonzero vector $v \in V$ is *cyclic* if $V = \{\rho(a)(v) : a \in A\}$.

   Show that (ρ, V) is irreducible if and only if all nonzero vectors in V are cyclic.

10. A left A-module V is *cyclic* if there exists a vector $v \in V$ with $V = \{av : a \in A\}$.

    Show that this occurs if and only if V is isomorphic to A/I for some left ideal in A.

11. Assume $\mathbf{k}$ has characteristic zero.

    Describe the finite-dimensional representations of the *Weyl algebra* $A = \langle x, y : yx - xy = 1 \rangle$.

12. Assume $\mathbf{k} = \mathbb{C}$ and let q be a nonzero complex number.

    Consider the q-*Weyl algebra* $A = \langle x, x^{-1}, y, y^{-1} : xy = qyx$ and $xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1\rangle$.

    Characterize the values of q such that A has finite-dimensional representations.

    Then describe all finite-dimensional irreducible representations of A for such q.