Instructions: Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due on **Tuesday, March 19**.

1. Let $p > 0$ be a prime integer and suppose G is a finite group of size p^n for some integer $n \geq 1$. Show that the every irreducible representation of G over a field $\mathbf{k}$ of characteristic p is *trivial* (meaning that all elements of G act as the identity transformation).

2. Show that if $\mathbf{k}$ is an algebraically closed field of positive characteristic $p > 0$ and G is a finite group whose size is divisible by p, then the number of isomorphism classes of irreducible representations of G over $\mathbf{k}$ is strictly less than the number of conjugacy classes in G.

3. Show that a finite group is abelian if and only if all of its irreducible representations over $\mathbb{C}$ are 1-dimensional.

4. The quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the group of order 8 with defining relations
$$i = jk = -kj, \quad j = ki = -ik, \quad k = ij = -ji, \quad \text{and} \quad -1 = i^2 = j^2 = k^2.$$
Explicitly work out the conjugacy classes and irreducible characters of Q_8 (over $\mathbb{C}$) to compute the group's character table. (In other words, fill in the details in Section 3.3 of the textbook. You can find the character table in the textbook, so make sure your computations match this!) Use the character table to find the irreducible decomposition of all products $\chi_1\chi_2$ for $\chi_1, \chi_2 \in \mathrm{Irr}(Q_8)$.

5. Let n be a positive integer. Define $I_2(n)$ to be the subgroup of $\mathrm{GL}_2(\mathbb{C})$ generated by
$$r = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
where $\theta = 2\pi/n$. Describe all irreducible complex representations of this group (up to isomorphism). You should consider the cases of odd and even n separately.

6. Let G be a finite group.

   Suppose $\chi_1, \chi_2, \ldots, \chi_r$ are the distinct irreducible character of G over $\mathbf{k} = \mathbb{C}$.

   For each $i \in \{1, 2, \ldots, r\}$ let
$$e_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \chi_i(g) g^{-1} \in \mathbb{C}[G].$$
Prove that these elements are central idempotents that sum to 1 in $\mathbb{C}[G]$. In other words, show that $1 = e_1 + e_2 + \cdots + e_r$ and $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$ and $e_i g = g e_i$ for all $g \in G$.

7. Let V be a finite dimensional complex vector space, and let $\mathrm{GL}(V)$ be the group of invertible linear transformations $V \to V$. Recall the definitions of the symmetric and exterior algebras
$$SV = \bigoplus_{n \geq 0} S^n V \quad \text{and} \quad \wedge V = \bigoplus_{n \geq 0} \wedge^n V.$$
Here $S^n V$ and $\wedge^n V$ are the images of $T^n V$ under the relevant quotient map.

   Both $S^n V$ and and $\wedge^n V$ are representations of $\mathrm{GL}(V)$ in a natural way.

   Show that $S^n V$ is irreducible for all $n \geq 0$ while $\wedge^n V$ is irreducible for $0 \leq n \leq \dim V$.

   Why is $\wedge^n V$ not an irreducible representation for $n > \dim V$?

8. The *adjacency matrix* of a graph Γ with n vertices (without multiple edges) is the matrix in which the entry in position (i, j) is 1 if the vertices i and j are connected with an edge, and zero otherwise.

   Let Γ be a finite graph whose automorphism group is nonabelian.

   Show that the adjacency matrix of Γ must have repeated eigenvalues.

9. Let G be a finite group with a complex representation V of finite dimension. Assume this representation is *faithful*, meaning that the corresponding map $G \to \mathrm{GL}(V)$ is injective. Show that each irreducible representation of G occurs a subrepresentation of $V^{\otimes n}$ for some $n > 0$.

10. Let $\mathbb{F}_q$ be a finite field with q elements. (Recall that we must have $q = p^k$ for some prime $p > 0$ and some integer $k > 0$.) Let G be the group of non-constant affine transformations $\mathbb{F}_q \to \mathbb{F}_q$ of the form $\lambda_{a,b} : x \mapsto ax + b$ for $0 \neq a \in \mathbb{F}_q$ and $b \in \mathbb{F}_q$.

    Find all irreducible complex representations of G, and compute their characters.

    Find the irreducible decomposition of all products $\chi_1\chi_2$ for irreducible characters $\chi_1, \chi_2 \in \mathrm{Irr}(G)$.