Instructions: Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due on **Tuesday, April 9**.

1. Let $G = \mathrm{SU}(2)$ be the group of unitary 2×2 matrices with determinant 1, where a square complex matrix A is unitary if $A^{-1} = \overline{A}^\top$. Let $V = \mathbb{C}^2$ be the vector space of 2-row column vectors with coefficients in $\mathbb{C}$, on which G acts by matrix multiplication.
   (a) Show that V is irreducible as a **real** representation of G.
   (b) Let $\mathbb{H} = \mathrm{End}_{\mathbb{R}[G]}(V)$ be the vector space of $\mathbb{R}$-linear maps $L : V \to V$ with $L(gv) = gL(v)$ for all $g \in G$ and $v \in V$. Show that $\mathbb{H}$ has an $\mathbb{R}$-basis consisting of elements $1, i, j, k$ where 1 is the identity map and the other elements satisfy
   $$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad \text{and} \quad ki = -ik = j.$$
   Deduce that $\mathbb{H}$ is closed under multiplication and that every nonzero element of $\mathbb{H}$ is invertible.
2. Let G be a finite group and let (V, ρ) be a complex representation of G that is *faithful* in the sense that $\rho : G \to \mathrm{GL}(V)$ is injective. Let χ be the character of (V, ρ). Show that any irreducible complex character ψ of G has $(\chi^n, \psi) > 0$ for some integer $n \geq 1$.
3. Suppose G is a finite group and V is a complex irreducible G-representation. Show that $\mathrm{End}_{\mathbb{R}[G]}(V)$ is isomorphic as an $\mathbb{R}$-algebra to $\mathbb{C}$ if V is of complex type, to the algebra $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ of real 2-by-2 matrices if V is of real type, and to $\mathbb{H}$ from Exercise #1 if V is of quaternionic type.
4. Suppose G is a nontrivial finite group with an odd number of elements. Show that G has an irreducible complex representation that is not of real type.
5. Suppose V is an irreducible complex representation of a finite group G with $\dim V > 1$. Show that there exists an element $g \in G$ with $\chi_V(g) = 0$, where χ_V is the character of V.
6. Suppose $K \subset H \subset G$ are groups and V is a representation of K. Show that $\mathrm{Ind}_H^G(\mathrm{Ind}_K^H(V))$ is isomorphic to $\mathrm{Ind}_K^G(V)$ as G-representations.
7. Suppose $H \subset G$ are finite groups and $\chi : H \to \mathbb{C} \setminus \{0\} = \mathrm{GL}_1(\mathbb{C})$ is a group homomorphism. Define
   $$e_\chi = \frac{1}{|H|} \sum_{h \in H} \chi(h) h^{-1} \in \mathbb{C}[H].$$
   Check that $e_\chi e_\chi = e_\chi$ and show that $\mathrm{Ind}_H^G(V) \cong \mathbb{C}[G]e_\chi$ as G-representations for $V = (\mathbb{C}, \chi)$.
8. Let $H \subset G$ be finite groups. Let V be a complex H-representation. Recall that the dual space V^* is then also an H-representation. Show that $\mathrm{Ind}_H^G(V^*) \cong (\mathrm{Ind}_H^G(V))^*$ as G-representations.
9. Find as explicit a formula as possible for $\sum_{\chi \in \mathrm{Irr}(S_n)} \chi(1)$ where n is a positive integer, S_n is the symmetric group of permutations of $\{1, 2, \ldots, n\}$, and $\mathrm{Irr}(S_n)$ is the set of irreducible complex characters of S_n.
10. In this problem we identify S_n with the group of bijections $\mathbb{Z} \to \mathbb{Z}$ that fix all integers $i \notin \{1, 2, \ldots, n\}$. Under this convention we automatically have $S_{n-1} \subset S_n \subset S_{n+1}$ for all n.

    Given a partition μ of a positive integer n, let $R(\mu)$ be the set of partitions of $n - 1$ whose *Young diagrams* (see `https://en.wikipedia.org/wiki/Young_tableau`) may be formed by removing exactly one cell from the Young diagram of μ, and let $A(\mu)$ be the set of partitions of $n + 1$ whose Young diagrams may be formed by adding exactly one cell to the Young diagram of μ.

    Let χ_μ be the character of S_n indexed by a partition μ of n. Show that
    $$\mathrm{Res}_{S_{n-1}}^{S_n}(\chi_\mu) = \sum_{\lambda \in R(\mu)} \chi_\lambda \qquad \text{and} \qquad \mathrm{Ind}_{S_n}^{S_{n+1}}(\chi_\mu) = \sum_{\lambda \in A(\mu)} \chi_\lambda.$$