Instructions: Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due on **Thursday, April 25**.

For the textbook problems, numbers refer to the online version of the book posted on the course website.

1. Problem 5.16.2 in the textbook.
2. Problem 5.16.3 in the textbook.
3. Show that if $s_\lambda(x_1, x_2, \dots, x_N)$ is the Schur polynomial of a partition λ in N variables, then
$$s_\lambda(1, z, z^2, \dots, z^{N-1}) = \prod_{1\leq i<j\leq N} \frac{z^{\lambda_i - i} - z^{\lambda_j - j}}{z^{-i} - z^{-j}}$$
and
$$s_\lambda(1, 1, 1, \dots, 1) = \prod_{1\leq i<j\leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$
4. Problem 5.24.1 in the textbook.
5. Read Section 5.25 in the textbook and then determine the character table of $\mathrm{GL}_2(\mathbb{F}_3)$.

   Give explicit representatives for each conjugacy class and describe the (complex) representations that give rise to each irreducible character. Clearly indicate how the rows and columns of your character table are labeled. You do not need to prove anything for this problem.
6. The terminology in this problem refers to Section 5.27 in the textbook.

   Fix a positive integer n. Let W_n be the group of permutations w of $\mathbb{Z}$ such that
$$w(-i) = -w(i) \text{ for all } i \in \mathbb{Z} \qquad \text{and} \qquad w(i) = i \text{ for all } i > n.$$
   Show that W_n is isomorphic to a *semidirect product* $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$.

   Specify explicitly the relevant homomorphism $\phi : S_n \to \mathrm{Aut}((\mathbb{Z}/2\mathbb{Z})^n)$.

   If λ is a partition then write $|\lambda| = \sum_i \lambda_i$ for the sum of its parts. A *bipartition* of n is an ordered pair (λ, μ) where λ and μ are partitions such that $|\lambda| + |\mu| = n$.

   Use the main theorem in Section 5.27 to show that there is a bijection from the set of complex irreducible characters of W_n to the set of bipartitions of n, such that if $\chi_{(\lambda,\mu)}$ is the character corresponding to a bipartition (λ, μ), then
$$\deg(\chi_{(\lambda,\mu)}) = \frac{n!}{|\lambda|!|\mu|!} \deg(\chi_\lambda) \deg(\chi_\mu).$$
   Here χ_λ and χ_μ are the characters of the Specht modules V_λ and V_μ, and if χ is a character then its degree is defined to be $\deg(\chi) = \chi(1)$.
7. Problem 6.1.1 in the textbook.
8. Compute the determinant of the matrix A_Γ for each of the Dynkin diagrams $\Gamma = A_N$, D_N, E_6, E_7, and E_8, as defined in Section 6.1 of the textbook.