Math 5112 - Lecture #12

Math 5112 - Lecture 12

Last time:

① character tables + applications, e.g. to decomposing

$$\chi_i \chi_j = \sum_k n_{ij}^k \chi_k$$

② Frobenius determinant of a finite group G:

let x_g for $g \in G$ be commuting indeterminants

then $$\det [x_{gh}]_{(g,h) \in G \times G} = \prod_{\psi \in \mathrm{Irr}(G)} P_\psi(x)^{\psi(1)}$$

(irreducible complex characters of G)

where each $P_\psi(x) \in \mathbb{C}[x_g \mid g \in G]$ is irreducible, of degree $\psi(1)$, and these factors are pairwise non-proportional

<u>Pf</u> Take $P_\psi(x) = \det\left(L(x) \mid \mathbb{C}[G]e_\psi\right)$

where $L(x) = \sum_{g \in G} x_g g \in \mathbb{C}[x][G]$

and $e_\psi = \frac{1}{|G|} \sum_{g \in G} \psi(g) g^{-1} \in \mathbb{C}[G]$. $\square$

Today and next few lectures:

[more topics related to reps of finite groups]

Bilinear forms

If V is (finite dim) vector space (over $\mathbb{C}$) then a bilinear form $(\cdot,\cdot): V \times V \to \mathbb{C}$ is the same thing as a linear map $L: V \to V^*$

$$(\cdot\,,\cdot) \longmapsto L: v \overset{\text{def}}{\mapsto} \big(w \mapsto (v,w)\big)$$

$$(v,w) \overset{\text{def}}{=} L(v)(w) \longmapsfrom L: V \to V^*$$

If V is a representation of a finite group G, then V^* is also a representation, and saying a form is <u>G-invariant</u> (meaning $(\rho_V(g)x, \rho_V(g)y) = (x,y)$ $\forall x,y \in V$)

is equivalent to saying the associated map $V \to V^*$ is a morphism of G-repns.

Saying a form is *nondegenerate* (meaning for each $x \in V \setminus 0$ there is $y \in V$ with $(x, y) \neq 0$) is equivalent to saying the associated map $V \to V^*$ is an isomorphism.

Prop If V is an irreducible repn of a finite group G, then a nonzero $\boxed{\text{G-invariant bilinear form}}$ $V \times V \to \mathbb{C}$ must be nondegenerate and the vector space of all G-invariant bilinear forms on V is zero or 1-dimensional.

Pf Schur's lemma: if V is irred. then so is V^*.
If $V \cong V^*$ as G-reps then $\mathrm{Hom}_G(V, V^*)$ is 1-dim, and otherwise it's zero.

Also, any morphism of G-reps $V \to V^*$ is either an isomorphism or zero. $\square$

doesn't require V to be
✓ a G-repn

Prop The space of bilinear forms on V is the direct sum of the subspaces of symmetric $((x,y)=(y,x))$ and skew-symmetric $((x,y)=-(y,x))$ forms on V.

Pf Any form $(\cdot,\cdot)$ is equal to

$(\cdot,\cdot)_{sym} + (\cdot,\cdot)_{ss}$ where

$$(v,w)_{sym} = \frac{1}{2}(v,w) + \frac{1}{2}(w,v)$$

$$(v,w)_{ss} = \frac{1}{2}(v,w) - \frac{1}{2}(w,v)$$

any only the zero form is both symmetric and skew-symmetric

$\square$

Cor If V is an irreducible G-repn where G is finite, then exactly one of the following holds:

① There is no G-invariant nondegenerate form on V

② There is a symmetric, G-invariant, nondeg. form on V

③ There is a skew-symmetric, G-invar., nondeg. form on V

Pf Space of all G-invariant forms on V is at most one dimensional and decomposes into the direct sum of skew symmetric and symmetric forms. □

Def.

If ① holds, then we say V is complex type

If ② holds, then we say V is real type

If ③ holds, then we say V is quaternionic type

[Fact: for quaternionic type reps V, $\mathrm{End}_{\mathbb{R}[G]}(V) \cong$ algebra $\mathbb{H}$ of quaternions]
May explore more on next HW

Let's state some conditions equivalent to ①, ②, ③ Assume V is an irreducible repn of a finite group G

Everything is defined over $\mathbb{C}$.

Prop V is of complex type iff any of the following equivalent properties hold:

(a) $\chi_V \neq \chi_{V^*}$

(b) $V \not\cong V^*$ as G-repns

(c) χ_V has values in $\mathbb{C} \setminus \mathbb{R}$

Pf Existence of a nondeg. G-invariant form on V is equivalent to (b), which is equiv to (a),

and (a) $\Leftrightarrow$ (c) because $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$. $\square$

Prop V is of real type iff in some basis of V, the matrices of $\rho(g)$ for all $g \in G$ have all real entries (in other words, V is realizable over $\mathbb{R}$)

[This implies that χ_V has all real values if V is real type]

Pf sketch Assume V is realizable over $\mathbb{R}$.

Let $v_1, v_2, \dots, v_n$ be basis with

$$\rho(g) v_i \in \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_n\} \quad \forall g \in G, \ 1 \leq i \leq n.$$

To get a nondegenerate, G-inv. sym form on V let $\langle \cdot, \cdot \rangle$ be positive def form with

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

then let $$(x, y) = \sum_{g \in G} \langle \rho_V(g) x, \rho_V(g), y \rangle$$

Since $\langle \cdot, \cdot \rangle$ is symmetric, positive definite (meaning $\langle x, x \rangle \geq 0$ with equality iff $x = 0$) same is true of $(\cdot, \cdot)$ which is clearly G-invariant.

Converse (that existence of such a form implies that V is realizable) is some more involved linear algebra ⇝ see next homework ∎

Prop V is of quaternionic type iff χ_V has all real values but V is not of real type.

Pf Latter conditions just mean that V is neither complex nor real type, so must be the only remaining type. $\square$

Frobenius-Schur indicator of V

Let $\varepsilon(V) = \varepsilon(\chi_V) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } V \text{ is real type} \\ 0 & \text{if } V \text{ is complex type} \\ -1 & \text{if } V \text{ is quaternionic type} \end{cases}$

Ex If $\mathbb{1}$ is trivial character $G \to \{1\}$

then $\varepsilon(\mathbb{1}) = 1$

If $G = \mathbb{Z}/n\mathbb{Z}$ and $\mathrm{sgn} : m \longmapsto (-1)^m$ (when n even)

then $\varepsilon(\mathrm{sgn}) = 1$ but all $\psi \in \mathrm{Irr}(\mathbb{Z}/n\mathbb{Z}) \setminus \{\mathbb{1}, \mathrm{sgn}\}$

have $\varepsilon(\psi) = 0$, since $\psi(m) = \zeta^m$ for some nth root of 1 in $\mathbb{C}$.

If $G = S_n$ symmetric group then $\varepsilon(\psi) = 1$ $\forall\, \psi \in \mathrm{Irr}(G)$

If $G = Q_8$ (see HW4) then there is a 2-d irr. repn

V with $\varepsilon(V) = -1$.

Thm $\boxed{\varepsilon(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)}$ for any $\chi \in \mathrm{Irr}(G)$

Cor $\boxed{\# \{ g \in G \mid g^2 = 1 \} = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\, \varepsilon(\chi)}$

call these the *involutions* of G

Pf of Cor

$$\# \{ g \in G \mid g^2 = 1 \} = \sum_{g \in G} \overbrace{\frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \chi(g^2)\chi(1)}^{= \begin{cases} 1 & g^2 = 1 \\ 0 & g^2 \neq 1 \end{cases} \text{ by char. orthog. relation}}$$

$$= \sum_{\chi \in \mathrm{Irr}(G)} \chi(1) \frac{1}{|G|} \sum_{g \in G} \chi(g^2) \underset{\substack{\uparrow \\ \text{by thm.}}}{=} \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\, \varepsilon(\chi) \quad \square$$

Pf of thm Let V be irred. G-repn with character χ_V. If $A: V \to V$ is any linear map with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (repeated with multipl.) then (by basic linear algebra)

$$\operatorname{trace}\left(A \otimes A|_{S^2 V}\right) = \sum_{i \leq j} \lambda_i \lambda_j$$

$$\operatorname{trace}\left(A \otimes A|_{\Lambda^2 V}\right) = \sum_{i < j} \lambda_i \lambda_j$$

so $\operatorname{trace}\left(A \otimes A|_{S^2 V}\right) - \operatorname{trace}\left(A \otimes A|_{\Lambda^2 V}\right) = \sum_i \lambda_i^2$

$= \operatorname{trace}(A^2)$

$\Rightarrow$ If $g \in G$ then

$$\chi_V(g^2) = \chi_{S^2V}(g) - \chi_{\Lambda^2V}(g)$$

Set $\pi = \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C}[G]$.

Then $\chi_V(\pi) = \dim \underbrace{V^G}_{}$

saw this last week

$= \{x \in V \mid \rho_V(g)x = x \ \forall g \in G]$

Thus $\frac{1}{|G|} \sum_{g \in G} \chi_V(g^2) = \frac{1}{|G|} \sum_{g \in G} \left(\chi_{S^2V}(g) - \chi_{\Lambda^2V}(g)\right)$

$= \chi_{S^2V}(\pi) - \chi_{\Lambda^2V}(\pi)$

$$= \underbrace{\overset{\leq 1}{\dim}\,(S^2V)^G - \overset{\leq 1}{\dim}\,(\Lambda^2V)^G}_{\text{Claim this is just } \varepsilon(V).}$$

To see claim, note:

$$(S^2V)^G \longleftrightarrow \text{symmetric } G\text{-invar. bilinear forms on } V$$

$$\sum_{i\leq j} (v_i,v_j)\, v_i\otimes v_j \longleftrightarrow (\cdot,\cdot)$$

if $\{v_i\}$ is basis of V

Similarly,

$$(\Lambda^2V)^G \longleftrightarrow \text{skew-sym } G\text{-invar. bilinear forms on } V$$

by same correspondence

thus either

$\dim(S^2V)^G = 1 = \varepsilon(V)$

$\dim(\Lambda^2V)^G = 0$

or

$\dim(S^2V)^G = 0$

$\dim(\Lambda^2V)^G = 1 = -\varepsilon(V)$

or

$\dim(S^2V)^G = 0 = \varepsilon(V)$

$\dim(\Lambda^2V)^G = 0$

$\square$

Algebraic numbers

Def $z \in \mathbb{C}$ is an algebraic number if it is a root of a polynomial in $\mathbb{Z}[x]$ (equivalently $\mathbb{Q}[x]$)

$z \in \mathbb{C}$ is an algebraic integer if it is a root of a monic polynomial in $\mathbb{Z}[x]$

↳ means leading term is 1 : $x^n +$ (lower degree terms)

Prop. $z \in \mathbb{C}$ is an algebraic number (integer) iff z is an eigenvalue of a square matrix with rational (integer) entries.

Pf Observe that any monic polynomial

$$p(z) = x^n + c_{n-1}x^{n-1} + \ldots + c_2x^2 + c_1x + c_0$$

is characteristic polynomial of matrix

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & & & 0 & & -c_1 \\ & 1 & & & & -c_2 \\ & & 1 & & & -c_3 \\ & & & \ddots & & \vdots \\ 0 & & & & 1 & -c_{n-1} \end{bmatrix}$$

□

Let $\overline{\mathbb{Q}}$ be set of algebraic numbers

Let $\mathbb{A}$ be set of algebraic integers.

Prop $\overline{\mathbb{Q}}$ is a field and $\mathbb{A}$ is a ring.
(subfield of $\mathbb{C}$) (subring of $\mathbb{C}$)

Pf If $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ has eigenvalue λ with eigenvector v and $B \in \mathrm{Mat}_{m\times m}(\mathbb{C})$ has eigenvalue μ with eigenvector w then

$$v \otimes w \text{ is eigenvector of } \begin{cases} A \otimes I_m \pm I_n \otimes B & \text{w/ eigenvalue } \lambda \pm \mu \\ A \otimes B & \text{w/ eigenvalue } \lambda\mu \end{cases}$$

Thus $\overline{\mathbb{Q}}$ and $\mathbb{A}$ are rings.

To see that $\overline{\mathbb{Q}}$ is a field note that if z is a nonzero root of $p(x) \in \mathbb{Z}[x]$ of degree n then $1/z$ is a root of $x^n p(\frac{1}{x}) \in \mathbb{Z}[x]$ $\square$

Prop. $\mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$

Pf Suppose z is root of $f(x) = x^n + c_{n-1}x^{n-1} + \dots + c_1 x + c_0$ and $z = p/q \in \mathbb{Q}$ where $p, q \in \mathbb{Z}$ with $\gcd(p,q) = 1$.

Then $0 = f(z) = f(p/q) = \dfrac{p^n + qk}{q^n}$ for some $k \in \mathbb{Z}$.

$\Rightarrow \quad p^n = -qk \in q\mathbb{Z}$ contradicting $\gcd(p,q)=1$

unless $q = \pm 1$ in which case $z \in \mathbb{Z}$. $\square$

Next time: Character values for a finite group are algebraic integers.