Math 5112 - Lecture #17

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Last time: constructed all complex irreducible repns of finite symmetric groups

Let n be a positive integer

Define S_n to be group of $\underbrace{\text{permutations}}_{\text{bijections } [n] \to [n]}$ of $[n] \overset{\text{def}}{=} \{1, 2, 3, \ldots, n\}$

For each partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0) \vdash n$ there is an associated Specht module V_λ (this is an S_n-repn / $\mathbb{C}$)

$$V_\lambda \stackrel{\text{def}}{=} \mathbb{C}[S_n]\, a_\lambda b_\lambda \quad \text{where}$$

$$a_\lambda = \sum_{\substack{g \text{ in row} \\ \text{stabilizer of } T_\lambda}} g \qquad b_\lambda = \sum_{\substack{g \text{ in column} \\ \text{stabilizer of } T_\lambda}} \operatorname{sgn}(g)\, g$$

called this subgrp $P_\lambda \subset S_n$ last time

called this subgroup $Q_\lambda \subset S_n$

for T_λ = (some fixed standard tableau of shape λ)

Last time we used T_λ defined such that $T_{(3,3,1,1)} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array}$

Thm Each V_λ for $\lambda \vdash n$ is an irreducible S_n-repn (over the field $\mathbb{C}$). Each irreducible S_n-repn (over the field $\mathbb{C}$) is isomorphic to some V_λ for a <u>unique</u> $\lambda \vdash n$.

Ex Trivial repn of S_n is $\cong V_\lambda$ for $\lambda = (n)$

Sign repn of S_n is $\cong V_\mu$ for $\mu = (1,1,1,\dots,1)$

(we'll see today that these are the only 1-d repns of S_n)

Rmk For any finite group, there is a bijection between the set of conjugacy classes and isomorphism classes of irreducible complex reprs (because the character table is square).

For S_n, one can specify this bijection in a concrete way:

$$\left\{ \begin{array}{c} \text{permutations of} \\ \text{cycle type } \lambda \end{array} \right\} \longleftrightarrow V_\lambda$$

Recall cycle type of, say, $\sigma = (1)(2\,5\,7\,8)(3\,10)(4\,9\,6)$ is $\lambda = (4,3,2,1)$

Aside — related to homework

Let G be a finite group with a complex finite dimensional G-repn (V, ρ)

There is a dual repn (V^*, ρ_{V^*}) and a conjugate repn $(\overline{V}, \overline{\rho})$. Here $\overline{V}$ is same as V but with modified scalar multiplication $c \cdot x \overset{\text{def}}{=} \overline{c}x$ for $c \in \mathbb{C}$, $x \in \overline{V}$

Both $\overline{V}$ and V^* have same character

$$\chi_{\overline{V}}(g) = \chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)} \text{ for } g \in G.$$

This means that $\bar{V} \cong V^*$.

The complexification of V is $V_{\mathbb{C}} \overset{\text{def}}{=} \mathbb{C} \otimes_{\mathbb{R}} V$

Elems of this space are sums of tensors of form $z \otimes_{\mathbb{R}} x$ for $z \in \mathbb{C}$, $x \in V$ where if $r \in \mathbb{R}$ then $zr \otimes_{\mathbb{R}} x = z \otimes_{\mathbb{R}} rx$.

$\mathbb{C}$ is a G-repn for trivial action $g : z \mapsto z \in \mathbb{C}$
V is a G-repn for the action $g : x \mapsto \rho(g)x$
Hence $V_{\mathbb{C}}$ is also a G-repn for (linear) action

$$g : z \otimes_{\mathbb{R}} x \mapsto z \otimes_{\mathbb{R}} \rho(g)x$$

Prop As a complex G-repn,

$$V_{\mathbb{C}} \cong V \oplus \overline{V} \cong V \oplus V^* \quad (\text{since } \overline{V} \cong V^*)$$

All proofs I've seen of this are quite abstract, so let's give a more explicit, constructive proof.

First, let's work through an example.

Ex Let $G = \mathbb{Z}/n\mathbb{Z} = \langle g \rangle$

Suppose $V = \mathbb{C}$ and $\rho(g) = \cos\theta + i\sin\theta$

(acts as a scalar)

where $\theta = \frac{2k\pi}{n}$, so that $\rho(g)^n = 1 \in \mathbb{C}$.

(for some $k \in \mathbb{Z}$)

Prop says that $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V \cong V \oplus \overline{V}$

so there should exist two 1-dimensional eigenspaces

for g acting on $V_{\mathbb{C}}$, one with eigenvalue

$\cos\theta + i\sin\theta$ and one with eigenvalue $\cos\theta - i\sin\theta$

What are these eigenspaces?

A basis for $V_{\mathbb{C}}$ is $\{1 \otimes_{\mathbb{R}} 1, 1 \otimes_{\mathbb{R}} i\}$

$$g \cdot (1 \otimes_{\mathbb{R}} 1) = 1 \otimes_{\mathbb{R}} (\cos\theta + i\sin\theta)$$
$$= \cos\theta\,(1 \otimes_{\mathbb{R}} 1) + \sin\theta\,(1 \otimes_{\mathbb{R}} i)$$

$$g \cdot (1 \otimes_{\mathbb{R}} i) = 1 \otimes_{\mathbb{R}} (i\cos\theta - \sin\theta)$$
$$= -\sin\theta\,(1 \otimes_{\mathbb{R}} 1) + \cos\theta\,(1 \otimes_{\mathbb{R}} i)$$

Thus $x \overset{\text{def}}{=} 1 \underset{\mathbb{R}}{\otimes} 1 - i \underset{\mathbb{R}}{\otimes} i$ and $y \overset{\text{def}}{=} 1 \underset{\mathbb{R}}{\otimes} 1 + i \underset{\mathbb{R}}{\otimes} i$

are eigenvectors for g with eigenvalues $\cos\theta \pm i\sin\theta$

So an isomorphism $V \oplus \overline{V} \xrightarrow{\sim} V_{\mathbb{C}}$ is given by

$$(z_1, z_2) \longmapsto z_1 x + z_2 y$$

$$= (z_1 + z_2) \otimes_{\mathbb{R}} 1 + (z_2 - z_1) i \otimes_{\mathbb{R}} i$$

<u>Pf of prop</u> Similarly, if $\{x_j\}_{j\in J}$ is a basis (over $\mathbb{C}$) for V

then a basis (over $\mathbb{C}$) for $V_{\mathbb{C}}$ is $\{1 \otimes_{\mathbb{R}} x_j, 1 \otimes_{\mathbb{R}} ix_j\}_{j\in J}$.

① The subspace spanned by $\{1 \otimes_{\mathbb{R}} x_j - i \otimes_{\mathbb{R}} ix_j\}_{j\in J}$ is $\cong V$ as G-repn

② The subspace spanned by $[1 \otimes_{\mathbb{R}} x_j + i \otimes_{\mathbb{R}} ix_j]_{j \in J}$ is $\cong \overline{V}$ as G-repn.

③ V_G is direct sum of these subrepns.

Explicitly: suppose $\rho(g) x_j = \sum_k (a_{jk} + i b_{jk}) x_k$ for some $a_{jk}, b_{jk} \in \mathbb{R}$ depending on g

Then:

$$g \cdot (1 \otimes_{\mathbb{R}} x_j + i \otimes_{\mathbb{R}} ix_j) = 1 \otimes_{\mathbb{R}} \rho(g) x_j + i \otimes_{\mathbb{R}} i \rho(g) x_j$$

$$= \sum_k (a_{jk} - i b_{jk})(1 \otimes_{\mathbb{R}} x_k) + \sum_k (b_{jk} - i a_{jk})(1 \otimes_{\mathbb{R}} i x_k)$$

$$= \sum_k (a_{jk} - i b_{jk})(1 \otimes_{\mathbb{R}} x_k + i \otimes i x_k)$$

But $\bar{\rho}(g)x_j = \sum_k (a_{jk} - i b_{jk}) x_k$

So this shows that $\mathbb{C}\text{-span}\{1 \otimes_{\mathbb{R}} x_j + i \otimes i x_j\}_{j \in J} \cong \bar{V}$

Likewise $\mathbb{C}\text{-span}\{1 \otimes_{\mathbb{R}} x_j - i \otimes_{\mathbb{R}} i x_j\}_{j \in J} \cong V$ as Grepn

$\square$

Some more results about S_n-repns (proofs sketched)

A cycle of $\sigma \in S_n$ is a set of form $\{\sigma^k(i) \mid k = 0, 1, 2, 3, \ldots\}$ for some $i \in \{1, 2, \ldots, n\}$. The set $\{1, 2, \ldots, n\}$ is a disjoint union of cycles of $\sigma \in S_n$, whose sizes arranged in order give the cycle type of σ.

Two permutations is S_n are conjugate if and only if they have same cycle type.

Let $i = (i_1, i_2, i_3, \dots)$ be a sequence of nonnegative integers with $n = \sum_{m \geq 1} m \cdot i_m$

Let C_i be a permutation in S_n with i_m cycles of size m for each $m = 1, 2, 3, \dots$

Ex If $n = 7$ and $i = (2, 1, 1, 0, 0, 0, \dots)$ then C_i could be $(1)(2)(34)(567)$

<u>Thm</u> (Frobenius character formula)

Choose a partition $\lambda \vdash n$ and let N be any integer with $N \geq \ell(\lambda)$ (the number of parts of λ) (can always take $N=n$)

then the character χ_λ of V_λ has value $\chi_\lambda(C_i)$ given by coefficient of

$$x^{\lambda+(N-1,N-2,\dots,3,2,1)} \stackrel{\text{def}}{=} \prod_{j=1}^{N} x_j^{\lambda_j+N-j}$$

in the polynomial $\displaystyle\prod_{1\leq j<k\leq N}(x_j-x_k)\prod_{m\geq 1}(x_1^m+x_2^m+\dots+x_N^m)^{i_m}$

Pf (sketch) Call the class function defined by this formula θ_λ. It's possible with some algebraic identifies to check that $\theta_\lambda(1) > 0$ and $(\theta_\lambda, \theta_\lambda) = 1$ so $\theta_\lambda \in \mathrm{Irr}(S_n)$ and hence equal to some χ_μ.

To show $\theta_\lambda = \chi_\lambda$, argue that θ_λ has a triangular $\mathbb{Z}$-linear combination expansion $\theta_\lambda = \chi_\lambda + (\text{terms } \chi_\mu \text{ with } \mu < \lambda \text{ in lex order})$

this part turns out to be zero

by expressing θ_λ as $\mathbb{Z}$-linear combination of certain induced characters $\mathrm{Ind}_{P_\lambda}^{S_n}(\mathbb{1})$ whose irr. decomp. are easy to understand. $\square$

Hook length formula

Let's use Frobenius character formula to compute

$$\chi_\lambda(1) = \dim V_\lambda .$$

Since $1 = C_i$ for $i = (n, 0, 0, 0, \dots)$

$\chi_\lambda(1)$ is equal to the coefficient of

(*) $\quad x_1^{\lambda_1+N-1} \, x_2^{\lambda_2+N-2} \cdots x_N^{\lambda_N}$

in product $\boxed{\displaystyle\prod_{1\le j<k\le N} (x_j - x_k)\,(x_1+x_2+\cdots+x_N)^n}$

$$\prod_{1\le j<k\le N} (x_j - x_k) = \det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ \vdots & & & & \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{bmatrix}$$

Thus, letting $\ell_j = \lambda_j + N - j$ so $(*) = x_1^{\ell_1} x_2^{\ell_2} \cdots x_N^{\ell_N}$,

$$x_\lambda(1) = \sum_{\substack{\sigma \in S_N \\ \ell_j \geq N - \sigma(j) \; \forall j \in [N]}} \operatorname{sgn}(\sigma) \frac{n!}{(\ell_1 - N + \sigma(1))!\,(\ell_2 - N + \sigma(2))! \cdots}$$

⋮

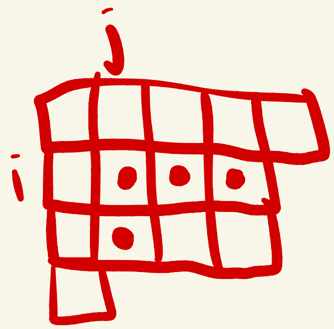
argue that this sum can be rewritten as

⋮

$$= \frac{n!}{\prod_{j=1}^{N} \ell_j!} \det\left[\ell_j(\ell_j - 1)(\ell_j - 2) \cdots (\ell_j - N + i + 1)\right]_{1 \leq i,j \leq N}$$

$$= \frac{n!}{\prod_{j=1}^{N} \ell_j!} \det\left[\ell_j^{N-i}\right]_{1 \leq i,j \leq N} = \frac{n!}{\prod_{j=1}^{N} \ell_j!} \prod_{1 \leq j < k \leq N} (\ell_j - \ell_k)$$

Define $h_\lambda(i,j) = \#$ of positions $(x,y) \in D_\lambda$ such that $(x = i \text{ and } j \le y)$ or $(x \ge i \text{ and } j = y)$

Ex If $D_\lambda =$  then $h_\lambda(i,j) = h_\lambda(2,2) = 4$

$\lambda = (5,4,4,1)$

Thm $\chi_\lambda(1) = \dim V_\lambda = \dfrac{n!}{\prod_{(i,j)\in D_\lambda} h_\lambda(i,j)}$

Pf (sketch) Check for each $x = 1, 2, 3, \dots$ that

$$\frac{\ell_x!}{\prod_{x<j\le N}(\ell_x-\ell_j)} = \prod_{k=1}^{\lambda_x} h_\lambda(x,k) \qquad \square$$

Ex Hook lengths for $\lambda = (5,4,4,1)$ are

8	6	5	4	1
6	4	3	2	
5	3	2	1	
1				

so $\dim V_{(5,4,4,1)} = \dfrac{14!}{8\cdot 6\cdot 6\cdot 5\cdot 5\cdot 4\cdot 4\cdot 3\cdot 3\cdot 2\cdot 2} = 7\cdot 7\cdot 13\cdot 11\cdot 3 = 21021$