Math 5112 - Lecture #18

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Last time:

① Let V be a finite dim. repn of a group G (defined over $\mathbb{C}$)

The <u>complexification</u> of V is

$$V_{\mathbb{C}} \overset{\text{def}}{=} \underset{\substack{\uparrow \\ \text{viewed} \\ \text{as trivial repn}}}{\mathbb{C}} \otimes_{\mathbb{R}} \underset{\substack{\uparrow \\ \text{viewed} \\ \text{as a } \mathbb{R}\text{-repn}}}{V} \underset{\substack{\uparrow \\ \text{as complex } G\text{-repns}}}{\cong} V \oplus V^{*} \overset{\substack{\text{requires } \dim V < \infty \\ \downarrow}}{\cong} V \oplus \overline{V}$$

$$V \cong \mathbb{C}\text{-span}\{1 \otimes_{\mathbb{R}} v - i \otimes_{\mathbb{R}} iv \mid \in V\} \subset V_{\mathbb{C}}$$

$$\bar{V} \cong \mathbb{C}\text{-span}\{1 \otimes_{\mathbb{R}} v + i \otimes_{\mathbb{R}} iv \mid v \in V\} \subset V_{\mathbb{C}}$$

② Frobenius character + hook length dim formulas (for irred. repns of symmetric group / $\mathbb{C}$)

Let $P_m(x) = x_1^m + x_2^m + \dots + x_N^m$ for each $m = 1, 2, 3, \dots$

Let $P_\mu(x) = \prod_{i=1}^{\ell(\mu)} P_{\mu_i}(x)$ for a partition $\mu = (\mu_1 \geq \mu_2 \geq \dots)$

Called the power sum symmetric polynomials.

Fix $\lambda \vdash n$ and $N \geq \ell(\lambda)$ = # of nonzero parts of λ

(a partition)

Thm The value of the irreducible character χ_λ of S_n at any permutation of cycle type $\mu \vdash n$ is the coefficient of the monomial

$$\prod_{j=1}^{N} x_j^{\lambda_j + N - j}$$

in the polynomial $\prod_{1 \leq i < j \leq N} (x_i - x_j) \cdot p_\mu(x)$

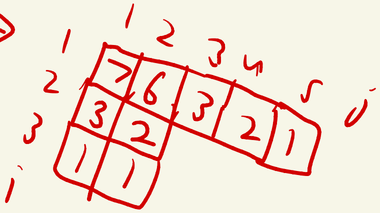
Thm Given $\lambda \vdash n$ let $h_{ij}(\lambda)$ be the hook length of position (i,j) in the Young diagram of λ.

Then

$$\boxed{\chi_\lambda(1) = \dim V_\lambda = \frac{n!}{\prod_{(i,j)} h_{ij}(\lambda)}}$$

best defined by example. If $\lambda = (5,2,2)$ then

$h_{ij}(\lambda) \leftrightarrow$



Today: Schur-Weyl duality

(A fundamental relationship between irred. repns of symmetric groups and general linear groups.)

Below, all algebras & repns are defined over an alg. closed field

Thm Let E be a finite dimension vector space.

Let A and B be subalgebras of $\mathrm{End}(E) = \{\text{linear maps } E \to E\}$

Assume A is semisimple and $B = \{b \in \mathrm{End}(E) \mid ab = ba \ \forall a \in A\}$

(In other words we assume $B = \mathrm{End}_A(E)$)

$= \left\{\begin{array}{l}\text{endomorphisms of } E \\ \text{as an } A\text{-repn.}\end{array}\right\}$

① $A = \{a \in \mathrm{End}(E) \mid ab = ba \ \forall b \in B\} = \mathrm{End}_B(E)$

In this situation we say that A and B are commuting algebras of each other.

② B is also semisimple

③ E is a repn of $A \otimes B$ for the linear action

$$a \otimes b : e \mapsto a(b(e)) = b(a(e)) \text{ for } \begin{matrix} a \in A \\ b \in B \end{matrix} \quad e \in E.$$

(this makes a repn because $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$)

There are irreducible A-repns $\{V_i\}_{i \in I}$ and irreducible B-repns $\{W_i\}_{i \in I}$ such that $E \overset{\text{as } A\otimes B\text{-repns}}{\cong} \bigoplus_{i \in I} V_i \otimes W_i$

④ Each irreducible repn of A (respectively, B) is isomorphic to V_i (respectively W_i) for a unique $i \in I$.

Thus, the map $V_i \leftrightarrow W_i$ for $i \in I$ gives a bijection between the isomorphism classes of irreducible A- and B-repns

Call this map the correspondence defined by E.

Pf Since A is semisimple, $E \cong \bigoplus_{i \in I} V_i \otimes W_i$ where

$\{V_i\}_{i \in I}$ represent all distinct isomorphism classes of irred. A-repns

and $W_i \overset{\text{def}}{=} \text{Hom}_A(V_i, E)$, and $A \cong \bigoplus_{i \in I} \text{End}(V_i)$

Once we make these identifications, Schur's lemma tells us that B (which is assumed to be the commuting algebra of A in $\mathrm{End}(E)$) is $\cong \bigoplus_{i \in I} \mathrm{End}(W_i)$.

This implies that B is semisimple.

Schur's lemma then implies A is the commuting algebra of B in $\mathrm{End}(E)$.

The remaining assertions are now clear from writing

$$A = \bigoplus_{i \in I} \mathrm{End}(V_i), \quad E = \bigoplus_{i \in I} V_i \otimes W_i, \quad B = \bigoplus_{i \in I} \mathrm{End}(W_i). \quad \square$$

Application: assume the ambient field is $\mathbb{C}$

Choose a (nonzero) finite-dim. $\mathbb{C}$-vector space V and let $n \in \{1,2,3,\dots\}$. Set $E = V^{\otimes n} = V \otimes \cdots \otimes V$

Let A be the image of $\mathbb{C}[S_n]$ in $\mathrm{End}(E)$ where the action of $\sigma \in S_n$ on E is by permuting tensor factors $\sigma : v_1 \otimes v_2 \otimes \cdots \otimes v_n \longmapsto v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$

Let $\mathfrak{gl}(V)$ be the Lie algebra of endomorphisms of V

$$\begin{array}{c}\mathfrak{gl}(V)\\ \shortparallel \\ \mathrm{End}(V)\end{array}$$

$\mathrm{End}(V)$ with $[x,y] = xy - yx$.

Finally let $B = \mathrm{End}_A(E) = \{ b \in \mathrm{End}(E) \mid ab = ba \ \forall a \in A \}$

<u>Thm</u> In this setting B is the image of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(V))$ in $\mathrm{End}(E)$ where the action of $g \in \mathfrak{gl}(V)$ on E is

$$g : v_1 \otimes v_2 \otimes \cdots \otimes v_n \longmapsto gv_1 \otimes gv_2 \otimes \cdots \otimes gv_n$$

($\mathcal{U}(\mathfrak{gl}(V)) \neq \mathfrak{gl}(V)$ although this is also an (associative unital) algebra containing the Lie algebra $\mathfrak{gl}(V)$)

Also, B is generated by elements of the form

$$\Delta_n(b) \overset{\text{def}}{=} b \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes b \otimes 1 \otimes \cdots \otimes 1$$

for $b \in \mathfrak{gl}(V)$

$$+ 1 \otimes 1 \otimes b \otimes 1 \otimes \cdots \otimes 1 + \cdots$$

$$+ 1 \otimes \cdots \otimes 1 \otimes b \in \mathrm{End}(V)^{\otimes n} \hookrightarrow \mathrm{End}(E)$$

as $E = V^{\otimes n}$

Pf Image of $\mathcal{U}(\mathfrak{gl}(V))$ is certainly contained in B.

Need to show that B is contained in image of $\mathcal{U}(\mathfrak{gl}(V))$.

We may identify $B = S^n(\mathrm{End}\, V)$ = span of the n-fold symmetric tensors of $\mathrm{End}(V)$

$S^n U$ is an irred. repn of $GL(V)$ for $U = \operatorname{End} V$ (or more generally for any finite dim. $\mathbb{C}$-vector space — HW exercise) and so it is spanned by elements of the form $u \otimes u \otimes \cdots \otimes u$ for $u \in U$

(since such elements span a nonzero subrepn)

(n-fold: 2-fold tensor product is $V \otimes V$
3-fold " " is $V \otimes V \otimes V$
4-fold " " is $V \otimes V \otimes V \otimes V$
etc.)

Thus B is spanned by $\{ b \otimes b \otimes \cdots \otimes b \mid b \in \mathfrak{gl}(V) \}$

But the elements $\Delta_n(b)$ for $b \in B$ generate this spanning set. Fundamental thm of symmetric functions (another exercise) says that there is a polynomial P with coeffs in $\mathbb{Q}$ such that

$$P(\Delta_n(b), \Delta_n(b^2), \dots \Delta_n(b^n)) = b \otimes b \otimes \cdots \otimes b$$

for each $b \in \mathfrak{gl}(V)$. Thus B is generated by the $\Delta_n(b)$ elements, each of which is contained in the image of $\mathcal{U}(\mathfrak{gl}(V))$, so B is also contained in this image. $\square$

The algebra $\mathbb{C}[S_n]$ is semisimple by Maschke's theorem, so A is also semisimple.

as $A \cong \mathbb{C}[S_n]$

Thus our first theorem implies:

Schur-Weyl duality ($\mathfrak{gl}(V)$ version)

① Images of $\mathbb{C}[S_n]$ and $\mathcal{U}(\mathfrak{gl}(V))$ in $\mathrm{End}(V^{\otimes n})$ are commuting algebras of each other, and both are semisimple.

② $V^{\otimes n}$ is a semisimple $\mathcal{U}(\mathfrak{gl}(V))$-module and $\mathbb{C}[S_n]$-module.

③ As $\mathbb{C}[S_n] \otimes \mathcal{U}(\mathfrak{gl}(V))$-modules, we have

$$\boxed{V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes L_\lambda}$$

where sum is over partitions of n,

V_λ is the Specht module for $\mathbb{C}[S_n]$ defined earlier, and each L_λ is either zero or an irreducible repn of $\mathfrak{gl}(V)$ and $L_\lambda \not\cong L_\mu$ if $\lambda \neq \mu$ and $L_\mu \neq 0$, $L_\lambda \neq 0$

The "duality" here refers to the correspondence $V_\lambda \Leftrightarrow L_\lambda$

Prop / Exercise Image of $GL(V) = \{ \text{invertible linear maps } V \to V \}$ in $\mathrm{End}(V^{\otimes n})$ spans $\mathrm{End}(V^{\otimes n})$

Pf sketch Want to show that any $b \otimes \cdots \otimes b$ for $b \in \mathfrak{gl}(V)$ is a linear comb of tensors $g \otimes \cdots \otimes g$ for $g \in GL(V)$. Can show this using fact that $\varepsilon I + b \in GL(V)$ for all sufficiently small $\varepsilon > 0$

Schur-Weyl duality (GL(V) version)

As a repn of $S_n \times GL(V)$, we have

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes L_\lambda$$

where $L_\lambda = \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n})$ are distinct non-isomorphic $GL(V)$-repns or zero.

Moral: Schur-Weyl duality gives a canonical correspondence between S_n-repns and $GL(V)$-repns which provides a consistent way of indexing both by partitions.