This document is a **transcript** of the lecture, so is more like an abbreviated set of lecture slides than complete lecture notes. For the latter, **consult the textbook** listed on the course webpage.

1 Associative Algebras

Setup: Let K be a field. Assume that K is algebraically closed unless noted otherwise.

Note that if K is algebraically closed, then every linear map $K^n \to K^n$ has an eigenvalue in K.

Usually we take $K = \mathbb{C}$, the complex numbers, or $K = \overline{\mathbb{F}_q}$, the algebraic closure of the finite field $\mathbb{F}_q$.

Definition 1.1. An *associative algebra* (over K) is a K-vector space A with a bilinear map $A \times A \to A$, written $(a, b) \mapsto a \cdot b$ or ab, that is associative in the sense that $a(bc) = (ab)c$ for all $a, b, c \in A$.

Here, “bilinear” means that the following properties hold for all $a, b, c \in A$ and $\lambda \in K$.

- $(a + b)c = ac + bc$
- $a(b + c) = ab + ac$
- $(\lambda a)b = a(\lambda b) = \lambda(ab)$

Because the product is associative, any way of parenthesizing an iterated product $a_1 a_2 a_3 \dots a_n$ with $a_i \in A$ gives the same result, so we can just omit the parentheses in such expressions.

Definition 1.2. A *unit* for an associative algebra A is an element $1 \in A$ with $1a = a1 = a$ for all $a \in A$.

Fact 1.3. If A has a unit then it is unique.

Proof. If 1 and $1'$ are units for A, then $1 = 11' = 1'$ since $a = a1'$ and $1a = a$. □

From now on, an *algebra* (over K) means a **nonzero**, **associative** algebra that **has a unit**. A *subalgebra* of an algebra is a subspace containing the unit that is closed under multplication.

Example 1.4. Let n be a positive integer. Here are some algebras:

- (*Trivial algebra*) The field K is itself an algebra. This is the smallest possible algebra, up to isomorphism, since the zero vector space is not an algebra.
- (*Polynomial algebra*) The set $K[x_1, x_2, \dots, x_n]$ of polynomials in commuting variables x_i with coefficients in K is an algebra with unit 1. This algebra is *commutative*, meaning $fg = gf$ for all elements f and g.
- (*Endomorphism algebra*) Let V be a K-vector space. Let $\operatorname{End} V$ be the vector space of K-linear maps $V \to V$. This is an algebra for the product given by composition $\rho_1\rho_2 \overset{\text{def}}{=} \rho_1 \circ \rho_2$ for $\rho_i : V \to V$. The unit is the identity map $\mathrm{id}_V : V \to V$.

  Aside: the vector space of *all* maps $V \to V$ is also algebra with the same product and unit, but this is an unreasonably high-dimensional object that is not of much interest.
- (*Free algebra*) The set $K\langle X_1, X_2, \dots, X_n\rangle$ of polynomials in non-commuting variables $X_1, X_2, \dots, X_n$ is also an algebra.
- (*Group algebra*) Given a group G. Let $K[G]$ be the K-vector space with basis $\{a_g : g \in G\}$. This becomes an algebra for the bilinear multiplication that has $a_g a_h = a_{gh}$ for $g, h \in G$. Unit is a_{1_G} where 1_G is unit for G.

Definition 1.5. A *morphism* $f : A \to B$ of algebras (over K) is a K-linear map such that

- $f(ab) = f(a)f(b)$ for $a, b \in A$.
- $f(1_A) = 1_B$.

We say f is an *isomorphism* if there exists a morphism $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. This occurs if and only if f is a bijection.

Example 1.6. There is a unique morphism $K\langle X_1, X_2, \ldots, X_n\rangle \to K[x_1, x_2, \ldots, x_n]$ that sends each variables $X_i \mapsto x_i$ (i.e., that lets the variables commute). In fact, if A is any algebra and we choose some elements $a_1, a_2, \ldots, a_n$, then there is a unique morphism $K\langle X_1, X_2, \ldots, X_n\rangle \to A$ sending each $X_i \mapsto a_i$.

Example 1.7. The field K viewed as a K-algebra is is *initial* in the category of K-algebras: there is a unique morphism $K \to A$ for any K-algebra A.

2 Representations

Definition 2.1. Let A be an algebra over K. A *representation* of A is a pair (ρ, V) where V is a K-vector space and $\rho : A \to \operatorname{End} V$ is an algebra morphism.

Notation. Sometimes we will say that the map "$\rho : A \to \operatorname{End} V$" is a representation. If ρ is known implicitly, we may also refer to V as a representation of A.

Definition 2.2. A *left A-module* is a vector space V with a bilinear map $A \times V \to V$, written $(a, v) \mapsto av$, such that

(1) $1_A v = v$ for all $v \in V$

(2) $a(bv) = (ab)v$ for $a, b \in A$, $v \in V$

Representations of A are the same as left A-modules in the following sense:

(1) If (ρ, V) is a representation, then setting

$$av \stackrel{\text{def}}{=} \rho(a)(v)$$

for $a \in A$, $v \in V$ makes V into a left A-module.

(2) If V is a left A-module, then setting

$$\rho(a)(v) \stackrel{\text{def}}{=} av$$

for $a \in A$, $v \in V$ defines a representation $\rho : A \to \operatorname{End} V$.

Moreover, operations (1) and (2) are inverses of each other.

Definition 2.3. Let A^{op} be the same vector space as A but with multiplication $a *_{\mathsf{op}} b = ba$ for $a, b \in A$. This gives another algebra with the same unit as A known as the *opposite algebra*.

It is instructive to check the associativity of $*_{\mathsf{op}}$ directly:

$$a *_{\mathsf{op}} (b *_{\mathsf{op}} c) = a *_{\mathsf{op}} (cb) = (cb)a = c(ba) = ba *_{\mathsf{op}} c = (a *_{\mathsf{op}} b) *_{\mathsf{op}} c.$$

Definition 2.4. A *right A-module* is a vector space V with a bilinear map $V \times A \to V$ written as $(v, a) \mapsto va$ for $a \in A$, $v \in V$ such that

(1) $v1_A = v$ for all $v \in V$.

(2) $(va)b = v(ab)$ for $a, b \in A$, $v \in V$.

Representations of A^{op} are the same as right A-modules, in the same sense as above.

If A is commutative, then $A = A^{\mathsf{op}}$. In this case, left A-modules are the same as right A-modules.

Example 2.5. Here are two common representations:

- (*Zero representation*) If $V = 0$, then $\operatorname{End} V$ consists of the unique map $0 \to 0$, and the unique map $A \to \operatorname{End} V = \{0 \to 0\}$ is an algebra morphism.
- (*Regular representation*) Define $\rho : A \to \operatorname{End} V$ by $\rho(a)(b) = ab$ for $a, b \in A$, then (ρ, A) is a representation of A.

If $A = K$, then any K-vector space is a left A-module and so affords a representation.

Definition 2.6. Suppose (ρ, V) is a representation of A. A *subrepresentation* of (ρ, V) is a subspace $W \subset V$ such that $\rho(a)(W) \subseteq W$ for all $a \in A$.

Note that 0 and V itself are always subrepresentations. We say that (ρ, V) is *irreducible* if $V \neq 0$ and there are no other subrepresentations.

Definition 2.7. If V is a left A-module, then a *submodule* is a subspace $W \subset V$ such that $aw \in W$ for all $a \in A$ and $w \in W$.

Under the correspondence between representations and left modules described above, subrepresentations correspond to submodules. In this sense, "subrepresentations are the same thing as submodules."

3 Morphisms of representations

Definition 3.1. Suppose (ρ_1, V_1) and (ρ_2, V_2) are representations of A.

A *morphism* $\phi : (\rho_1, V_1) \to (\rho_2, V_2)$ is a linear map $\phi : V_1 \to V_2$ such that

$$\phi(\rho_1(a)(v)) = \rho_2(a)(\phi(v))$$

for all $a \in V$ and $v \in V_1$. This property holds precisely when the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\phi} & V_2 \\ {\scriptstyle \rho_1(a)}\downarrow & & \downarrow{\scriptstyle \rho_2(a)} \\ V_1 & \xrightarrow{\phi} & V_2 \end{array}$$

commutes for all $a \in A$.

We say that ϕ is an isomorphism if ϕ is a bijection.

Proposition 3.2 (*Schur's Lemma*)**.** For this result, K may be any field, not necessarily algebraically closed. Let (ρ_1, V_1) and (ρ_2, V_2) be representations of A. Suppose $\phi : (\rho_1, V_1) \to (\rho_2, V_2)$ is a nonzero morphism.

(1) If (ρ_1, V_1) is irreducible then ϕ is injective.

(2) If (ρ_2, V_2) is irreducible then ϕ is surjective.

(3) If both representations are irreducible then ϕ is an isomorphism.

Proof. Check that $\ker \phi = \{v \in V_1 : \phi(v) = 0\} \subset V_1$ and $\mathrm{Im}\phi = \{\phi(v) : v \in V_1\} \subset V_2$ are subrepresentations. But $\ker \phi \neq V_1$ and $\mathrm{Im}\phi \neq 0$ if ϕ is nonzero. The result therefore follows since only 0 and V can be subrepresentations of an irreducible representation (ρ, V). $\square$

For the last two results we go back to assuming that K is algebraically closed.

Corollary 3.3. Assume K is algebraically closed and (ρ, V) is an irreducible representation of A with V finite dimensional. Suppose $\phi : (\rho, V) \to (\rho, V)$ is a morphism. Then there exists a scalar $\lambda \in K$ such that $\phi(v) = \lambda v$ for all $v \in V$, that is, $\phi = \lambda \cdot \mathrm{id}_V$ is a scalar map.

Proof. As K is algebraically closed, there must be an eigenvalue for ϕ, i.e. there must be some $\lambda \in K$ such that $\phi - \lambda \cdot \mathrm{id}_V$ is not invertible. But $\phi - \lambda \cdot \mathrm{id}_V$ is another morphism $(\rho, V) \to (\rho, V)$. So we must have $\phi - \lambda \cdot \mathrm{id}_V = 0$ by Schur's Lemma. □

Corollary 3.4. If K is algebraically closed and A is commutative, then every irreducible representation (ρ, V) of A has $\dim V = 1$

Proof. If (ρ, V) is a representation, then the map $\rho(a) : V \to V$ for any $a \in A$ is a morphism $(\rho, V) \to (\rho, V)$ since A is commutative. By Corollary 2.10, we must have $\rho(a) = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in K$.

But this applies to every $a \in A$, so every subspace of V is a subrepresentation.

Therefore V is irreducible if and only if $\dim V = 1$. □