This document is a **transcript** of the lecture, so is more like an abbreviated set of lecture slides than complete lecture notes. For the latter, **consult the textbook** listed on the course webpage.

1 Review from last time

Let K be a field, assumed to be algebraically closed unless noted otherwise. (E.g., $K = \mathbb{C}$ or $\overline{\mathbb{F}_p}$)

Definition 1.1. An *associative algebra* (over K) is a K-vector space A with a bilinear map $A \times A \to A$, written $(a, b) \mapsto ab$, that satisfies $a(bc) = (ab)c$ for all $a, b, c \in A$.

Definition 1.2. When we refer to an *algebra*, we will always mean an associative algebra that is **nonzero** and has a **unit element** 1 satisfying $1a = a1 = a$ for all $a \in A$.

This means that if A is an algebra then $0 \neq 1$ in A. The zero vector space is not an algebra.

Definition 1.3. A *morphism* $f : A \to B$ of algebras (over K) is a K-linear map such that

- $f(ab) = f(a)f(b)$ for $a, b \in A$
- $f(1_A) = 1_B$

We say f is an *isomorphism* if f is a bijection.

Example 1.4. If V is a vector space (over K), then $\mathrm{End}(V) \overset{\mathrm{def}}{=} \{\text{linear maps } V \to V\}$ is an algebra, where the product is composition $\rho_1\rho_2 \overset{\mathrm{def}}{=} \rho_1 \circ \rho_2$, and the unit is $\mathrm{id}_V : V \to V$.

Definition 1.5. A *representation* of an algebra A is a pair (ρ, V) where V is a K-vector space and $\rho : A \to \mathrm{End}\, V$ is an algebra morphism.

Often we identify (ρ, V) with the left A-module structure on V in which A acts by

$$a \cdot v \overset{\mathrm{def}}{=} \rho(a)(v) \quad \text{for } a \in A,\ v \in V.$$

Definition 1.6. A *morphism* $\phi : (\rho_1, V_1) \to (\rho_2, V_2)$ is a linear map $\phi : V_1 \to V_2$ such that $\phi(\rho_1(a)(v)) = \rho_2(a)(\phi(v))$ for all $a \in A$, $v \in V_1$. (Sometimes called an *intertwining operator*)

We say that ϕ is an isomorphism if ϕ is a bijection of vector spaces.

Definition 1.7. A *subrepresentation* of (ρ, V) is a subspace $W \subseteq V$ with $\rho(a)(W) \subseteq W$ for all $a \in A$.

If W is a subrepresentation, then it makes sense to say that the pair (ρ, W) is a representation, where we reinterpret ρ as a map $A \to \mathrm{End}(W)$ by taking appropriate restrictions.

We say that (ρ, V) is *irreducible* if $V \neq 0$ and there are no other subrepresentations except V and 0.

Proposition 1.8 (*Schur's Lemma*)**.** Let $\phi : (\rho_1, V_1) \to (\rho_2, V_2)$ be a morphism of representations.

(a) If both representations are irreducible then ϕ is an isomorphism.

(This holds even when K is not algebraically closed)

(b) If $(\rho, V) = (\rho_1, V_1) = (\rho_2, V_2)$ and this representation is finite dimensional ($\dim V < \infty$) and irreducible, and K is algebraically closed, then ϕ is a **scalar map**, so $\phi = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in K$.

(c) If K is algebraically closed and A is *commutative* ($ab = ba$ for all $a, b \in A$) then every irreducible representation (ρ, V) has $\dim V = 1$.

Example 1.9 (Important counterexamples)**.** Assume $K = \mathbb{R}$ is the field of real numbers, which not algebraically closed. Then (b) and (c) can both fail, in the following way:

Let $A \overset{\text{def}}{=} \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\} \overset{\text{def}}{=} V$.

As $\mathbb{R}$-algebras, $A \cong \mathbb{C}$. Let $\rho : A \to \mathrm{End}(V) = \mathrm{End}(A)$ be the regular representation, i.e. $\rho(y)(z) \overset{\text{def}}{=} yz$.

Every $0 \neq z \in A$ is invertible, so (ρ, V) is irreducible with (real) dimension 2. This "contradicts" (c) above since A is commutative.

Define $\phi\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right) = \begin{bmatrix} -b & -a \\ a & -b \end{bmatrix}$, which is multiplication by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

This is a morphism $\phi : (\rho, V) \to (\rho, V)$ since A is commutative, but it is not a scalar map for the scalars $K = \mathbb{R}$, "contradicting" (b).

2 Indecomposable representations

Let A be an algebra over K, not necessarily algebraically closed. Suppose (ρ_1, V_1) and (ρ_2, V_2) are representations of A. Then we can form the *direct sum representation*

$$(\rho_1, V_1) \oplus (\rho_2, V_2) \overset{\text{def}}{=} (\rho_1 \oplus \rho_2, V_1 \oplus V_2),$$

where $(\rho_1 \oplus \rho_2)(a)(v_1 + v_2) \overset{\text{def}}{=} \rho_1(a)(v_1) + \rho_2(a)(v_2)$ for $a \in A$, $v_1 \in V_1$ and $v_2 \in V_2$, and $V_1 \oplus V_2$ is a direct sum of vector spaces.

Note that $(\rho_1, V_1) \oplus (\rho_2, V_2) \cong (\rho_2, V_2) \oplus (\rho_1, V_1)$.

Definition 2.1. A representation (ρ, V) is *indecomposable* if it is not isomorphic to $(\rho_1, V_1) \oplus (\rho_2, V_2)$ for any nonzero representations (ρ_i, V_i). This occurs if and only if (ρ, V) does not have two nonzero subrepresentations $W_1, W_2 \subset V$ with $V = W_1 \oplus W_2$ as any internal direct sum.

Notation. If $W_1, W_2 \subseteq V$ are subspaces then writing

(a) $V = W_1 \oplus W_2$

is just an abbreviation for

(b) it holds that $V = W_1 + W_2$ and $0 = W_1 \cap W_2$.

In general, the direct sum $W_1 \oplus W_2$ is some new vector space satisfying a universal property, with canonical inclusions of W_1 and W_2. When (b) holds, the ambient vector space V satisfies these conditions so can be identified with $W_1 \oplus W_2$.

Note that irreducible $\implies$ indecomposable, but not vice versa.

Example 2.2. Consider the (commutative) polynomial algebra $A = K[x]$. What are the irreducible representations of A?

Choose a linear map $L : V \to V$, where V is a vector space.

Define $\rho_L : K[x] \to \mathrm{End}(V)$ by formula such that

$$\rho_L(f(x)) = f(L) \iff \rho_L(a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0) = a_n L^n + \ldots + a_2 L^2 + a_1 L + a_0 I.$$

Then (ρ_L, V) is a representation of $K[x]$.

Every representation of $K[x]$ must arise via this construction because every algebra morphism $A \to B$ is uniquely determined by the image of the variable x. It is possible that different choices of L might give isomorphic representations (ρ_L, V), however.

The representation (ρ_L, V) is irreducible if and only if $\dim V = 1$ since K is algebraically closed and $K[x]$ is commutative.

What are the indecomposable representations of $K[x]$?

Choose $\lambda \in K$ and an integer $n \geq 1$. Define $J_{\lambda,n} : K^n \to K^n$ to be the linear map with matrix

$$\begin{bmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{bmatrix}.$$

For example, $J_{\lambda,3} =$ linear map with matrix $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$.

Then $(\rho_{J_{\lambda,n}, K^n})$ is indecomposable (but not irreducible if $n > 1$), and every indecomposable representation of $K[x]$ is isomorphic to one of these representation, by the uniqueness of Jordan canonical form.

Moreover, it holds that $(\rho_{J_{\lambda,n}}, K^n) \cong (\rho_{J_{\lambda',n'}}, K^{n'})$ if and only if $n = n'$ and $\lambda = \lambda'$.

These statements are not self-evident; their proofs requires a lot of linear algebra work.

3 Group Representations

Suppose G is a group. Given a vector space V, let $\mathrm{GL}(V)$ be the group of invertible linear maps $V \to V$.

Definition 3.1. A *group representation* of G is a pair (ρ, V) where V is a vector space and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism.

Group representations are the same as representations of the corresponding group algebra.

Recall that the group algebra is $K[G] = K\text{-span}\{a_g : g \in G\}$ where $a_g a_h = a_{gh}$.

We can turn any group representation (ρ, V) for G into a representation of $K[G]$ by setting $\rho(a_g) = \rho(g)$ and extending by linearity.

Conversely, if (ρ, V) representation of $K[G]$ then every $\rho(a_g) \in \mathrm{GL}(V)$ for $g \in G$ is invertible and

$$g \mapsto \rho(a_g) \in \mathrm{GL}(V)$$

is a group homomorphism $G \to \mathrm{GL}(V)$. This holds since for every invertible $a \in A$ in any algebra, we have $\rho(a)\rho(a^{-1}) = \rho(1_A) = \mathrm{id}_V$ for any representation (ρ, V).

4 Ideals in algebras

Let A be an algebra.

Definition 4.1. A *left ideal* in A is a subspace $I \subset A$ with $aI \overset{\text{def}}{=} \{ai : i \in I\} \subseteq I$ for all $a \in A$.

A *right ideal* in A is a subspace $I \subset A$ with $Ia \subset I$ for all $a \in A$.

A *two-sided ideal* in A is a subspace that is both a left and right ideal.

All three notions coincide if A is commutative.

Left ideals are the same as subrepresentations of the regular representation of A and right ideals are the same as subrepresentations of the regular representation of A^{op}.

The subspaces 0 and A are always two-sided ideals. If these are the only two-sided ideals then A is *simple*.

Example 4.2. The algebra $\mathrm{Mat}_{n\times n}(K)$ is simple. To check this, we need to show that if $I \subseteq \mathrm{Mat}_{n\times n}(K)$ is a nonzero two-sided ideal then every $n \times n$ matrix is in I. If there is some elementary matrix $E_{jk} \in I$, then every other elementary matrix is obtained as $E_{il} = E_{ij}E_{jk}E_{kl} \in I$ so any linear combination of elementary matrices is in I, which means that every $n \times n$ matrix is in I. So it is enough to show that I contains some elementary matrix. As I is nonzero, there is some $0 \neq M \in I$ with some nonzero entry $M_{jk} \neq 0$, and then we have $E_{jk} = \frac{1}{M_{jk}} E_{jj} M E_{kk} \in I$ as needed.

Example 4.3. If $\phi : A \to B$ is an algebra morphism then the *kernel*

$$\ker\phi \stackrel{\text{def}}{=} \{a \in A : \phi(a) = 0\}$$

is a two-sided ideal. The kernel is always a subspace, and if $\phi(a) = 0$ then $\phi(xay) = \phi(x)\phi(a)\phi(y) = 0$ for all $x, y \in A$. Taking $x = 1$ shows that A is right ideal and taking $y = 1$ shows that A is a left ideal, so it is a two-sided ideal.

Example 4.4. If $S \subset A$ is any set, then we define $\langle S\rangle$ to be the intersection of all two-sided ideals in A containing S. We call this the **two-sided ideal generated by** S. Exercise: show that all elements of $\langle S\rangle$ have the form $a_1s_1b_1 + \ldots + a_ns_nb_n$ for some $n \geq 0$ and some $a_i, b_i \in A$, $s_i \in S$.

Example 4.5. A *maximal* left/right/two-sided ideal $I \subsetneq A$ is an ideal properly contained in exactly one other left/right/two-sided ideal (namely A itself). One can use Zorn's lemma to show that every ideal is contained in a maximal ideal. (Zorn's lemma is only needed if A is infinite-dimensional.)

Assume I is two-sided ideal in an algebra A with $I \neq A$. Then the quotient vector space

$$A/I = \{a + I : a \in A\}$$

where $a + I \stackrel{\text{def}}{=} \{a + i : i \in I\}$ is an algebra for the multiplication defined by

$$(a + I)(b + I) = ab + I \quad \text{for } a, b \in A.$$

The unit is $1 + I$. There is something to check to make sure that the above multiplication is well-defined. This is a standard exercise. The linear map $\pi : A \to A/I$ with $\pi(a) = a + I$ is an algebra morphism.

Definition 4.6. If (ρ_V, V) is a representation of A and $W \subset V$ is a subrepresentations, then we define the $\rho_{V/W} : A \to \mathrm{End}(V/W)$ by the formula

$$\rho_{V/W}(a)(x + W) = \rho_V(a)(x) + W \text{ for } a \in A, x \in V.$$

Then $(\rho_{V/W}, V/W)$ is a representation of A, called the *quotient representation*.

If $I \subseteq A$ is a left ideal, then A/I is a representation of A via this construction.

Equivalently, A/I is a left A-module for the action $a \cdot (b + I) \stackrel{\text{def}}{=} ab + I$ for $a, b \in A$.

5 Generators and relations

Recall that $K\langle X_1, \ldots, X_n\rangle$ is the *free algebra* of polynomials in noncommuting variables.

If $f_1, \ldots, f_m \in K\langle X_1, \ldots, X_n\rangle$ then we can consider the quotient algebra

$$K\langle X_1, X_2, \ldots, X_n\rangle/\langle\{f_1, f_2, \ldots, f_m\}\rangle,$$

which we often denote by writing

$$K\langle X_1, X_2, \ldots X_n \mid f_1 = f_2 = \ldots f_m = 0\rangle.$$

We think of the elements of this quotient are polynomials as usual, but we can replace expressions equal to f_i by zero.

Remark 5.1. Technically, if $I = \langle\{f_1, f_2, \ldots, f_m\}\rangle$ then elements of $K\langle X_1, X_2, \ldots, X_n\rangle/I$ are cosets of the form $f + I$. Usually we write things by dropping the "$+I$" part, even though this can make it ambiguous whether f belongs to $K\langle X_1, X_2, \ldots, X_n\rangle$ or the quotient.

Example 5.2. The *Weyl algebra* is

$$K\langle x, y \mid yx - xy - 1 = 0\rangle = K\langle x, y \mid yx - xy = 1\rangle.$$

In the Weyl algebra, we have $yx = xy + 1$ and $xyx = x(xy + 1) = x^2y + x = (yx - 1)x = yx^2 - x$.

Example 5.3. The *q-Weyl algebra* for a fixed nonzero element $q \in K$ is

$$K\left\langle x, x^{-1}, y, y^{-1} \mid yx = qxy \text{ and } xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1\right\rangle.$$

The second set of relations ensures that x, x^{-1} and y, y^{-1} are inverses of each other.