This document is a **transcript** of the lecture, so is more like an abbreviated set of lecture slides than complete lecture notes. For the latter, **consult the textbook** listed on the course webpage.

1 Review from last time

Let A be an algebra over a field K, which is algebraically closed unless otherwise noted.

Given two representations (ρ_1, V_1) and (ρ_2, V_2) of A, we can form the *direct sum representation*

$$(\rho_1, V_1) \oplus (\rho_2, V_2) \overset{\text{def}}{=} (\rho_1 \oplus \rho_2, V_1 \oplus V_2)$$

where $(\rho_1 \oplus \rho_2)(a)(v_1 + v_2) = \rho_1(a)(v_1) + \rho_2(a)(v_2)$ for $v_1 \in V_1$, $v_2 \in V_2$, $a \in A$.

Definition 1.1. A representation (ρ, V) for A is *indecomposable* if it is not isomorphic to any direct sum $(\rho_1, V_1) \oplus (\rho_2, V_2)$ with $V_1 \neq 0$ and $V_2 \neq 0$.

Clearly NOT indecomposable $\implies$ NOT irreducible, since if $(\rho, V) \cong (\rho_1, V_1) \oplus (\rho_2, V_2)$ then each (ρ_i, V_i) corresponds to a subrepresentation.

Taking contrapositives, this means that irreducible $\implies$ indecomposable, but not vice versa.

Definition 1.2. A representation (ρ, V) is *semisimple* or *completely reducible* if we have

$$(\rho, V) \cong \bigoplus_{i \in I} (\rho_i, V_i)$$

where each (ρ_i, V_i) is irreducible.

Given any subspace $I \subseteq A$, let $A/I = \{a + I : a \in A\}$ be the vector space quotient.

A subspace I is a *left/right/two-sided ideal* in A if $AI \subset I$ or $IA \subset I$ or $AIA \subset I$, respectively.

If I is a **proper** two-sided ideal then A/I is an algebra, with product $(a+I)(b+I) = ab + I$ for $a, b \in A$.

(Note that if $I = A$ then I is a two-sided ideal but $A/I = 0$, which we do not consider to be an algebra.)

If I is a left/right ideal then A/I is naturally a left/right A-module.

If (ρ, V) is an A-representation and $W \subset V$ is a subrepresentation, then we can form a quotient representation $(\rho, V/W)$ by interpreting $\rho : A \to \mathrm{End} V$ as a map $A \to \mathrm{End}(V/W)$ via formula

$$\rho(a)(v + W) = \rho(a)(v) + W \quad \text{for } v \in V.$$

Remark 1.3 (Issues with quotients)**.** One must check that formulas involving cosets $x + S$ (where x is an element and S is a set) give the same result if we replace x by any y such that $x + S = y + S$. Usually what needs to be checked is routine, but we tend to omit the details.

Quotient algebras are useful since they let us define algebras by generators and relations.

Example 1.4. The *Weyl algebra* is

$$A = \langle x, y \mid yx - xy = 1 \rangle.$$

This means that A is the quotient of free algebra $K\langle x, y \rangle$ by the two sided ideal

$$\langle yx - xy - 1 \rangle \overset{\text{def}}{=} \{\text{intersection of all two-sided ideals containing } yx - xy - 1\}.$$

To save space, we write $f(x,y)$ instead of $f(x,y) + \langle yx - xy - 1\rangle$ to denote elements of A.

It often hard to say concretely what an ideal like $\langle yx - xy - 1\rangle$ is explicitly, and to classify precisely which expressions in $\langle x, y\rangle$ become zero in the quotient. In practice, taking quotients means we can make substitutions like $yx = xy + 1$ in polynomial expressions.

The relations in an algebra defined by generators and relations provide an algorithm for transforming a given expression to others that are equal in the algebra. In principle, an exhaustive search using this algorithm can tell you if two expressions are equal, but this search might not terminate.

2 Weyl algebra

Things work out nicely for the Weyl algebra $A = K\langle x, y \mid yx = xy + 1\rangle$.

Proposition 2.1. A basis for the Weyl algebra is $\{x^i y^j : i, j \geq 0\}$.

Proof. It is easy to see that the set spans algebra, since

$$x^{i_1} y^{j_1} x^{i_2} y^{j_2} \ldots x^{i_k} y^{j_k} = x^{i_1+i_2+\ldots+i_k} y^{j_1+j_2+\ldots+j_k} + (\text{lower degree terms})$$

via repeated substitutions $yx = xy + 1$.

To show linear independence, assume $\mathsf{char}(K) = 0$.

(The argument when $\mathsf{char}(K) > 0$ is similar but not as elegant; see the textbook for details.)

Consider the polynomial ring $K[z]$. For $f \in K[z]$, define $x \cdot f = zf$ and $y \cdot f = \frac{df}{dz}$.

There is a unique left A-module structure on $K[z]$ with these formulas, because

$$y \cdot (x \cdot f) = y \cdot (zf) = \frac{d}{dz}(zf) = f + z\frac{df}{dz} = f + x \cdot (y \cdot f),$$

which is equivalent to $yx = xy + 1$.

Now suppose $c_{ij} \in K$ are such that $\sum_{i,j} c_{ij} x^i y^j = 0$ in A.

Let $L = \sum_{i,j} c_{ij} z^i \left(\frac{d}{dz}\right)^j$ be a differential operator on $K[z]$. Then

$$L(f) = \left(\sum_{i,j} c_{ij} x^i y^j\right) \cdot f = 0 \text{ for all } f \in K[z].$$

But we can write $L = \sum_{j=0}^{r} Q_j(z) \left(\frac{d}{dz}\right)^j$ for some polynomials $Q_j(z) \in K[z]$.

Now observe that

$$\begin{aligned}
L(1) &= Q_0(z) = 0 \\
L(z) &= Q_0(z)z + Q_1(z) = Q_1(z) = 0 \\
L(z^2) &= Q_0(z)z^2 + Q_1(z)z + Q_2(z) = Q_2(z) = 0 \\
&\vdots
\end{aligned}$$

Thus we have $Q_0 = Q_1 = \ldots = Q_r = 0 \implies c_{ij} = 0$ for every i, j, which proves that elements $x^i y^j$ must be linearly independent. $\square$

Example 2.2. The *q-Weyl algebra* is

$$A = K\langle x, x^{-1}, y, y^{-1} \mid yx = qxy, xx^{-1} = x^{-1}x = 1, yy^{-1}y^{-1}y = 1\rangle.$$

Here, $q \in K$ is a fixed nonzero element.

We require q to be nonzero, since if $q = 0$ then $x = y = 0$ so $A = 0$:

$$yx = 0 \implies y^{-1}yx = x = 0 \text{ and } yxx^{-1} = y = 0.$$

Proposition 2.3. If $q \neq 0$, then a basis for the q-Weyl algebra is $\{x^i y^j : i, j \in \mathbb{Z}\}$

Proof. The argument to show that give elements span the algebra is similar to the Weyl algebra case. For linear independence, see the textbook. □

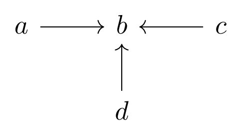
3 Quiver representations

Quivers are another source of algebra representations.

Definition 3.1. A *quiver* $Q = (I, E)$ is a directed graph with self-loops and multiple edges allowed. Here, I is the set of vertices in Q and E is the multi-set of directed edges $i \to j$.

(A *multi-set* is, informally, a set allowing repeated elements. This can be viewed formally as a **map** from an arbitrary set to the set of positive integers $\{1, 2, 3, \dots\}$.)

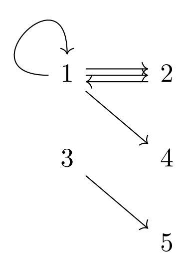
Example 3.2. We draw the quiver $Q = (\{a, b, c, d\}, \{a \to b, c \to b, d \to b\})$ as



Example 3.3. The quiver $Q = (I, E)$, where $I = \{1, 2, 3, 4, 5\}$ and

$$E = \{1 \to 1, 1 \to 2, 1 \to 2, 2 \to 1, 1 \to 4, 3 \to 5\},$$

can be drawn as



Definition 3.4. A *representation* (V_*, ρ_*) of a quiver $Q = (I, E)$ is an assignment of a vector space V_i for each $i \in I$ and a linear map $\rho_{ij} : V_i \to V_j$ for each edge $i \to j$ in E.

Why is this relevant? Quiver representations are natural to consider because they contain the same data as an arbitrary diagram of linear maps between vector spaces. Moreover, there is a related *path algebra* whose algebra representations are in bijection with quiver representations.

Definition 3.5. The path algebra Path_Q of a quiver $Q = (I, E)$ is the K-vector space with a basis given by all directed paths in Q, including trivial paths p_i for each $i \in I$, with multiplication of paths given by

$$(i_0 \to \imath_1 \to \ldots \to i_n) \cdot (j_0 \to j_1 \to \ldots \to j_m) \overset{\text{def}}{=} \begin{cases} j_0 \to j_1 \to \ldots \to j_m \to i_1 \to i_2 \ldots i_n & \text{if } j_m = i_0 \\ 0 & \text{if } j_m \neq i_0 \end{cases}$$

If Q has a finite set of vertices, then the element $\sum_{i \in I} p_i$ is the unit in Path_Q.

We can translate between representations of Q and Path_Q:

(1) Suppose (ρ, V) is a representation of Path_Q.

Define a representation of Q by setting $V_i = \rho(p_i)(V)$ for $i \in I$ and

$$\rho_{ij} = \rho(i \to j)|_{V_i} : V_i \to V_j$$

for edges $i \to j$ in E.

The definition of ρ_{ij} makes sense as a map $V_i \to V_j$ since

$$\begin{aligned} \rho(i \to j)(V_i) &= \rho(i \to j) \circ \rho(p_i)(V) = \rho(i \to j \cdot p_i)(V) = \rho(i \to j)(V) \\ &= \rho(p_j \cdot i \to j)(V) = \rho(p_j) \circ \rho(i \to j)(V) \subset \rho(p_j)(V) = V_j. \end{aligned}$$

(2) Suppose (V_*, ρ_*) is a quiver representation of Q. Form a representation of Path_Q by setting

$$V = \bigoplus_{i \in I} V_i$$

and let $\rho(i_0 \to i_1 \to \ldots i_m)$ be the unique linear map $V \to V$ that sends

$$\begin{cases} V_j = V_{i_0} \xrightarrow{\rho_{i_0 i_1}} V_{i_1} \xrightarrow{\rho_{i_1 i_2}} \ldots \xrightarrow{\rho_{i_{m-1} i_m}} V_{i_m} & \text{if } i_0 = j \\ V_j \to 0 & \text{if } i_0 \neq j. \end{cases}$$

In particular, $\rho(p_i)$ is the projection $V \to V_i$. Then (ρ, V) is a representation of Path_Q.

The operations (1) and (2) are inverses of each other.

We have notions of *direct sums*, *subrepresentations*, *irreducibility*, *indecomposability* and *morphisms* for quiver representations. The definitions are similar to the algebra representation case. See the textbook for precise formulations.

4 Lie algebra representations

Lie algebras are another sources of "representations" that can be viewed as a special case of algebra representations. Despite the name, Lie algebras are not "algebras" according to our definition, since their products are not associative. Here is the actual definition:

Let $\mathfrak{g}$ be a vector space over K.

Assume $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear map satisfying $[a, a] = 0$ for all $a \in \mathfrak{g}$.

This property implies that $[a, b] = -[b, a]$ for all $a, b \in \mathfrak{g}$, since

$$0 = [a + b, a + b] = [a, a + b] + [b, a + b] = [a, a] + [a, b] + [b, a] + [b, b] = [a, b] + [b, a].$$

Definition 4.1. We say that $\mathfrak{g}$ is a *Lie algebra* relative to the *bracket* $[\cdot,\cdot]$ if the **Jacobi identity** holds:

$$[[a,b],c]+[[b,c],a]+[[c,a],b]=0 \text{ for all } a,b,c\in\mathfrak{g}.$$

Example 4.2. If A is any associative algebra, then $[a,b]\stackrel{\text{def}}{=}ab-ba$ makes A into a Lie algebra.

Example 4.3. Let $\mathsf{Der}(A)$ be a vector space of linear maps $D:A\to A$ (for an algebra A) satisfying $D(ab)=aD(b)+D(a)b$. This is a Lie algebra for the bracket $[D_1,D_2]=D_1\circ D_2-D_2\circ D_1$.

Example 4.4. If V is any vector space then we write $\mathfrak{gl}(V)$ for the *general linear Lie algebra* obtained by giving the vector space $\mathrm{End}(V)$ the bracket $[f,g]=f\circ g-g\circ f$. This is a special case of Example 4.2.

A *Lie subalgebra* of a Lie algebra is a subspace closed under the bracket.

Theorem 4.5 (Ado's Theorem)**.** If $\mathfrak{g}$ is a finite-dimensional Lie algebra, then $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional vector space V.

Definition 4.6. A *morphism* of Lie algebras is a linear map $\phi:\mathfrak{g}_1\to\mathfrak{g}_2$ such that

$$\phi([a,b])=[\phi(a),\phi(b)] \text{ for all } a,b\in\mathfrak{g}.$$

Definition 4.7. A *representation* of a Lie algebra $\mathfrak{g}$ is a pair (ρ,V) where V is a vector space and $\rho:\mathfrak{g}\to\mathfrak{gl}(V)$ is a Lie algebra morphism.

Example 4.8. The *adjoint representation* of $\mathfrak{g}$ is $(\rho,\mathfrak{g})$ where $\rho(a)(b)=[a,b]$ for $a,b\in\mathfrak{g}$.

For any Lie algebra $\mathfrak{g}$, there is a related algebra, called the *universal enveloping algebra* $\mathfrak{U}(\mathfrak{g})$, such that there is a bijective correspondence between the Lie algebra representations of $\mathfrak{g}$ and the algebra representations of $\mathfrak{U}(\mathfrak{g})$.

If $\mathfrak{g}$ has a basis $\{x_i\}_{i\in I}$ and $c_{ij}^k\in K$ are the coefficients such that $[x_i,x_j]=\sum_k c_{ij}^k x_k$ for each $i,j\in I$, then $\mathfrak{U}(\mathfrak{g})$ may be defined via generators and relations as the quotient algebra

$$\mathfrak{U}(\mathfrak{g})=K\left\langle x_i \text{ for } i\in I \mid x_ix_j-x_jx_i=\sum_k c_{ij}^k x_k \text{ for all } i,j\in I\right\rangle.$$