This document is a **transcript** of the lecture, so is more like an abbreviated set of lecture slides than complete lecture notes. For the latter, **consult the textbook** listed on the course webpage.

1 Review from last time

In the previous lecture we introduced two more sources of representations in representation theory:

- *Quivers* $Q = (I, E)$, which are directed graphs with vertices I and edges E, with self-loops and multiple edges allowed. A *quiver representation* $(V_\bullet, \rho_\bullet)$ consists of the following data:
  1. For each vertex $i \in I$, we assign a vector space V_i.
  2. For each (directed) edge $i \to j$, we assign a linear map $\rho_{ij} : V_i \to V_j$.

  There is a bijection between quiver representations of Q and algebra representations of its associated *path algebra*, which is unital when I is finite.

- *Lie algebras* $(\mathfrak{g}, [\cdot, \cdot])$, which consist of a K-vector space $\mathfrak{g}$ and bilinear map $[\cdot, \cdot]$ called the *Lie bracket* satisfying:
  1. **Anticommutativity:** $[x, x] = 0$ for all $x \in \mathfrak{g}$, which implies $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.
  2. **Jacobi identity:** We have $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ $\forall x, y, z \in \mathfrak{g}$.

  Despite the name, a Lie algebra is not technically an algebra, at least how we have defined it, since when we say "algebra" we implicitly assume associativity and the existence of a unit element, but the Lie bracket for a Lie algebra usually has neither property.

  A *Lie algebra representation* is a pair (V, ρ) such that
  1. V is a vector space over the same ground field K as $\mathfrak{g}$.
  2. $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is Lie algebra morphism, meaning a linear map preserving the relevant Lie brackets. Here $\mathfrak{gl}(V)$ is the set of all linear maps $V \to V$ (so the same vector space as $\mathrm{End}(V)$) but viewed as a Lie algebra with bracket
  $$[L_1, L_2] = L_1 \circ L_2 - L_2 \circ L_1, \ \forall L_1, L_2 \in \mathfrak{gl}(V).$$

  There exists a bijection between Lie algebra representation of $\mathfrak{g}$ and algebra representations of $U(\mathfrak{g})$, its *universal enveloping algebra* of $\mathfrak{g}$ (which will be given a second definition later today).

2 Tensor products of vector spaces

Let V and W be two K-vector spaces. Their direct product is simply the set of pairs

$$V \times W = \{(v, w) : v \in w, \ b \in W\}.$$

This object is just a set, not a vector space. Define the *free product* $V * W$ to be the K-vector space with $V \times W$ as a basis. Each element of $V * W$ is a finite linear combination of pairs $(v, w) \in V \times W$.

One way to define the *tensor product* of V and W is as the quotient vector space

$$V \otimes W \overset{\text{def}}{=} (V * W)/\mathcal{I}_{V,W}$$

where $\mathcal{I}_{V,W}$ is the subspace spanned by all elements of the form

- $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$,
- $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$,

- $(av, w) - a(v, w)$, or
- $(v, aw) - a(v, w)$,

for any $a \in K$, $v_1, v_2, v \in V$, and $w_1, w_2, w \in W$.

If $x \in V$ and $y \in W$, then we write $x \otimes y \in V \otimes W$ for the image of the pair $(x, y) \in V \times W \subset V * W$ under the quotient map $V * W \to V \otimes W$. This means that

$$x \otimes y \stackrel{\text{def}}{=} (x, y) + \mathcal{I}_{V,W}$$

if we view elements of a vector space quotient a cosets of a subspace.

We refer to $x \otimes y$ as a *pure tensor*. Not all elements of $V \otimes W$ are pure tensors, but every element is a finite linear combination of pure tensors.

We can manipulate pure tensors without changing their value in $V \otimes W$ using the following identities:

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \quad v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, \quad (cv) \otimes w = c(v \otimes w) = v \otimes (cw)$$

for $v_1, v_2, v \in V$, $w_1, w_2, w \in W$, and $c \in K$.

These equations hold because the differences between the two sides belong to the subspace $\mathcal{I}_{V,W}$.

This means that we can have $x \otimes y = x' \otimes y'$ when $x \neq x'$ and $y \neq y'$.

A simple example is when $x' = -x \in V$ and $y' = -y \in W$.

Exercise 2.1 (Important to do once)**.** If $\{v_i : i \in I\}$ is a basis of V and $\{w_j : j \in J\}$ is a basis of W then the set of pure tensors $\{v_i \otimes w_j : (i, j) \in I \times J\}$ is a basis of $V \otimes W$.

Exercise 2.2. If U, V, and W are K-vector spaces, then there is a unique isomorphism

$$(U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$$

that sends $u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w$ for each $u \in U$, $v \in V$, and $w \in W$.

As a result of this exercise, there is a canonical isomorphism between any way of forming the tensor product between a finite sequence of vector spaces (n principle, each way requires us to choose a parenthesization of the factors, since we can only tensor two spaces at a time). For example:

$$V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \cong V_1 \otimes (V_2 \otimes (V_3 \otimes V_4)) \cong (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \cong ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \cong (V_1 \otimes (V_2 \otimes V_3)) \otimes V_4.$$

In view of this, we will ignore the issue of parenthesization and just define

$$V^{\otimes 0} \stackrel{\text{def}}{=} K \quad \text{and} \quad V^{\otimes n} \stackrel{\text{def}}{=} V \otimes \cdots \otimes V \ (n \text{ factors}).$$

3 Tensor products of linear maps

If $f \in \operatorname{Hom}(V, V')$ and $g \in \operatorname{Hom}(W, W')$ are two linear maps then their *tensor product* is the unique linear map $f \otimes g : V \otimes W \to V' \otimes W'$ that acts on pure tensors as

$$v \otimes w \mapsto f(v) \otimes g(w) \quad \text{for all } v \in V \text{ and } w \in W.$$

There are some things to check to make sure that this is well-defined. Since $V \times W$ is a basis for $V * W$, there is certainly a unique linear map $f * g : V * W \to V' \otimes W'$ that sends

$$(v, w) \mapsto f(v) \otimes g(w) \quad \text{for all } v \in V \text{ and } w \in W.$$

We want know that the map $f * g$ descends to a well-defined map of quotient spaces $V \otimes W \to V' \otimes W'$, since this will give exactly our desired map $f \otimes g$. So we need to verify that $(f * g)(\mathcal{I}_{V,W}) = 0 \subseteq V' \otimes W'$.

To check this, it is enough to show that $f * g$ sends each element in the spanning set for $\mathcal{I}_{V,W}$ to zero. This is some fairly routine algebra. For instance, if $v_1, v_2 \in V$ and $w \in W$ then we have

$$\begin{aligned}(f * g)((v_1 + v_2, w) - (v_1, w) - (v_2, w)) &= (f * g)((v_1 + v_2, w)) - (f * g)((v_1, w)) - (f * g)((v_2, w)) \\ &= f(v_1 + v_2) \otimes g(w) - f(v_1) \otimes g(w) - f(v_2) \otimes g(w) \\ &= f(v_1) \otimes g(w) + f(v_2) \otimes g(w) - f(v_1) \otimes g(w) - f(v_2) \otimes g(w) \\ &= 0\end{aligned}$$

as needed. The calculations showing that $f * g$ kills off the other elements spanning $\mathcal{I}_{V,W}$ are similar.

4 Tensor algebra

In this section, we introduce an object called the *tensor algebra* of a vector space V.

This is given as a vector space by the infinite direct sum

$$\mathcal{T}V \stackrel{\text{def}}{=} \bigoplus_{n \geq 0} V^{\otimes n}.$$

Remember that the elements of an infinite direct sum are finite sums of elements from the summands.

We view $\mathcal{T}V$ as a K-algebra by defining

$$ab \stackrel{\text{def}}{=} a \otimes b \quad \text{for } a \in V^{\otimes m} \text{ and } b \in V^{\otimes n},$$

and extending by bilinearity. Here we view $a \otimes b \in V^{\otimes (m+n)}$. This product is associative, since the tensor product is associative. The unit of the resulting *tensor algebra* $\mathcal{T}V$ is the field unit $1 = 1_K \in K = V^{\otimes 0}$.

Notice that $\mathcal{T}V$ is an algebra even when $V = 0$, since then $\mathcal{T}V = \mathcal{T}0 = K$.

Exercise 4.1. We may identify tensor algebras with free algebras. Suppose V is finite-dimensional with basis $\{v_1, \dots, v_N\}$. Then there is a unique algebra isomorphism

$$\mathcal{T}V \xrightarrow{\sim} K\langle X_1, \dots, X_N \rangle$$

that sends $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \mapsto X_{i_1} X_{i_2} \cdots X_{i_k}$. A similar isomorphisms exists when V is infinite-dimensional, if we allow infinitely-many variables in the free algebra.

We mention three interesting quotients of the tensor algebra.

4.1 Symmetric algebras

The first quotient is called the *symmetric algebra* of V. This is defined by

$$\mathcal{S}V \stackrel{\text{def}}{=} \mathcal{T}V / \langle v \otimes w - w \otimes v : v, w \in V \rangle.$$

Recall that "$\langle v \otimes w - w \otimes v : v, w \in V \rangle$" means the intersection of all two-sided ideals in $\mathcal{T}V$ containing all of the differences $v \otimes w - w \otimes v$ for each $v, w \in V$.

The symmetric algebra $\mathcal{S}V$ is always commutative. We have $\mathcal{T}V \cong \mathcal{S}V$ if and only if $\dim V \leq 1$.

Example 4.2. We may identify symmetric algebras with polynomial algebras. Suppose V is finite-dimensional with basis $\{v_1, \dots, v_N\}$. Then there is a unique algebra isomorphism

$$\mathcal{S}V \xrightarrow{\sim} K[x_1, \dots, x_N]$$

that sends $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \mapsto x_{i_1} x_{i_2} \cdots x_{i_k}$. A similar isomorphisms exists when V is infinite-dimensional, if we allow infinitely-many variables in the polynomial algebra.

4.2 Exterior algebras

The second quotient is called the *exterior algebra* of V. This is defined by

$$\bigwedge V \stackrel{\text{def}}{=} \mathcal{T}V/\langle v\otimes v : \ v \in V\rangle.$$

Define $x \wedge y$ to be the image $x \otimes y \in V \otimes V$ under the quotient map $\mathcal{T}V \to \bigwedge V$. Then

$$\begin{aligned} 0 &= (x+y)\wedge(x+y) \\ &= x\wedge y + x\wedge y + y\wedge x + y\wedge y \\ &= x\wedge y + y\wedge x \end{aligned}$$

so $x \wedge y = -y \wedge x$. This shows that the operation $\wedge$ defines an *anti-commutative* product for $\bigwedge V$.

Example 4.3. Choosing a basis for V determines an isomorphism from $\bigwedge V$ to a "polynomial algebra" in which the variables anti-commute in the sense that $x_i x_j = -x_j x_i$.

4.3 Universal enveloping algebras

If $\mathfrak{g}$ is a Lie algebra then its *universal enveloping algebra* is the quotient of the tensor algebra

$$U(\mathfrak{g}) \stackrel{\text{def}}{=} \mathcal{T}\mathfrak{g}/\langle x\otimes y - y \otimes x - [x,y] : x,y \in \mathfrak{g}\rangle.$$

This definition is equivalent to the one in the last lecture. The advantage of this formulation is that it does not depend on a choice of basis for V. Our previous definition relied on such a choice, and it was not clear that we got the same algebra for different choices of basis.

5 Tensor product of modules

Building on our definition of vector space tensor products, we can now define more general tensor products of modules over a (not necessarily commutative) algebra.

5.1 Right modules with left modules

Consider the following setting:

1. A, B, C are algebras over the same field K.
2. V is a right B-module.
3. W is a left B-module.

Then we define $V \otimes_B W$ to be the vector space quotient

$$V \otimes_B W \stackrel{\text{def}}{=} (V \otimes W)/K\text{-span}\{vb\otimes w - v\otimes bw : v \in V, w \in W, b \in B\}.$$

In general, this object only has the structure of a K-vector space.

Specifically, if B is non-commutative, then $V \otimes_B W$ is not naturally a left or right module for B.

We refer to $V \otimes_B W$ as the *tensor product of V and W over B*. If $v \in V$ and $w \in W$ then we write

$$v \otimes_B w \in V \otimes_B W$$

for the image of $v \otimes w \in V \otimes W$ under the quotient map $V \otimes W \to V \otimes_B W$. Notice that if $b \in B$ then

$$vb \otimes_B w = v \otimes_B bw.$$

5.2 Bimodules

Continuing the setup from the previous section, we now assume in addition that:

- V is an (A, B)-*bimodule*, meaning that
  1. V has **both** right B-module and left A-module structures;
  2. these structures are compatible in the sense that $(av)b = a(vb)$ for all $a \in A, b \in B, v \in V$.

Assume likewise that

- W is a (B, C)-*bimodule*, meaning that
  1. W has **both** left B-module and right C-module structures;
  2. these structures are compatible in the sense that $(bw)c = b(wc)$, for all $b \in B, c \in C, w \in W$.

Then the vector space $V \otimes_B W$ has a (A, C)-bimodule structure defined by

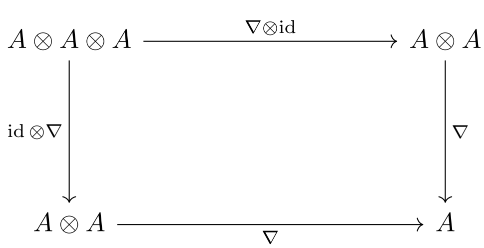
$$\begin{cases} a(v \otimes_B w) \stackrel{\text{def}}{=} (av) \otimes_B w & \text{if } a \in A \\ (v \otimes_B w)c \stackrel{\text{def}}{=} v \otimes_B (wc) & \text{if } c \in C \end{cases} \quad \text{for } v \in V \text{ and } w \in W.$$

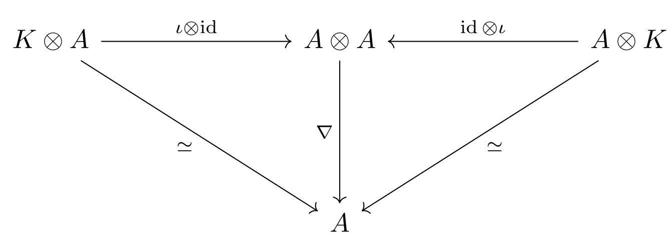
The case when $A = B = C$ is worth noting. In this situation, V and W are both (B, B)-bimodules, and the tensor product $V \otimes_B W$ is also a (B, B)-bimodule.

Remark 5.1. If the algebra B is commutative, then left and right B-modules are the same as (B, B)-bimodules (do you see why?), and so we **can** form the tensor product of two left B-modules or two right B-modules. However, this is secretly just doing the (B, B)-bimodule tensor product.

6 Diagrammatic definition of an algebra

Now that we have a good handle on vector space tensor products, we can given an alternate definition of an *algebra*. This consists of a K-vector space A with linear maps $\nabla : A \otimes A \to A$ and $\iota : K \to A$ that make the following diagrams commute:





The diagonal arrows on the right are the linear maps $K \otimes A \to A$ and $A \otimes K \to A$ sending $1_K \otimes a \mapsto a$ and $a \otimes 1_K \mapsto a$ for all $a \in A$. These maps are vector space isomorphisms.

Under this formulation, the product in A is $ab \stackrel{\text{def}}{=} \nabla(a \otimes b)$ and the unit is $\iota(1_K) \in A$.

One nice feature of this definition is that it naturally suggests the definition of a *coalgebra*: this is the object one gets by repeating the above definition but reversing the direction of all arrows.