This document is a **transcript** of the lecture, so is more like an abbreviated set of lecture slides than complete lecture notes. For the latter, **consult the textbook** listed on the course webpage.

1 Review from last time

1.1 Tensor products of vector spaces

In the last lecture, we defined tensor products of vector spaces over fields and algebras.

We briefly review the definition for vector spaces V and W over an arbitrary field K.

- The *direct product* $V \times W$ is the set of pairs (v, w) with $v \in V$ and $w \in W$.
- The *free product* $V * W$ is the K-vector space with $V \times W$ as a basis.
- The *tensor product* $V \otimes W$ is the quotient vector space $V \otimes W \overset{\text{def}}{=} (V * W)/\mathcal{I}_{V,W}$ where $\mathcal{I}_{V,W}$ is the subspace of $V * W$ spanned by the elements of the following forms:
  1. $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$,
  2. $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$,
  3. $(av, w) - a(v, w)$,
  4. $(v, aw) - a(v, w)$,

  for any $a \in K, v_1, v_2, v \in V$ and $w_1, w_2, w \in W$.

The image of $(v, w) \in V \times W$ under the quotient map $V * W \to V \otimes W$ is denoted $v \otimes w \in V \otimes W$ and called a *pure tensor*. For any $a \in K, v_1, v_2, v \in V$, $w_1, w_2, w \in W$ it holds that

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \quad v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, \quad av \otimes w = v \otimes aw = a(v \otimes w).$$

Fact. If V has basis $\{v_i\}_{i \in I}$ and W has basis $\{w_j\}_{j \in J}$ then $\{v_i \otimes w_j\}_{(i,j) \in I \times J}$ is a basis for $V \otimes W$.

1.2 Tensor algebras and their quotients

Because the tensor product is associative in the sense that we can identify $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$, we can define iterated tensor products $V^{\otimes 0} \overset{\text{def}}{=} K$ and $V^{\otimes n} := V \otimes V \otimes \cdots \otimes V$ (n terms) when V is any K-vector space. Using these notations, the *tensor algebra* of V is defined as

$$\mathcal{T}V \overset{\text{def}}{=} \bigoplus_{n \geq 0} V^{\otimes n}.$$

This is an (associative, unital) algebra with product $xy \overset{\text{def}}{=} x \otimes y$ and unit $1 \in K = V^{\otimes 0} \subset \mathcal{T}V$.

Notice that if $V = 0$ then $\mathcal{T}V = K$, and that $\mathcal{T}V$ is commutative if and only if $\dim V \leq 1$.

Any choice of basis for V determines an isomorphism from $\mathcal{T}V$ to a free algebra $K\langle X_1, X_2, \dots \rangle$.

Three notable quotients of $\mathcal{T}V$:

1. The *symmetric algebra* of V is $\mathcal{S}V \overset{\text{def}}{=} \mathcal{T}V/\langle v \otimes w - w \otimes v : v, w \in V \rangle$.
2. The *exterior algebra* is $\bigwedge V \overset{\text{def}}{=} \mathcal{T}V/\langle v \otimes v : v \in V \rangle$.
3. The *universal enveloping algebra* of V, when V is a Lie algebra with bracket $[\cdot, \cdot]$, is

$$\mathcal{U}V \overset{\text{def}}{=} \mathcal{T}V/\langle v \otimes w - w \otimes v - [w, v] : v, w \in V \rangle.$$

2 Semisimple representations

In this lecture, we begin a new chapter focusing on some general results about algebra representations.

2.1 Basic definitions and examples

From now on, we will assume that K is an algebraically closed field, and A is a K-algebra.

Definition 2.1. A representation of A is *semisimple* or *completely reducible* if it is isomorphic to a direct sum of irreducible representations.

As a general rule in mathematical terminology:

"simple" ≡ "irreducible" and "semisimple" ≡ "(direct) sum of simple objects".

Notation. Suppose V is a left A-module. Often we will say that "V is a representation of A": this just means the representation (V, ρ) where $\rho : A \to \mathrm{End}(V)$ is defined by $\rho(a) : x \mapsto ax$ for $a \in A$ and $x \in V$.

Example 2.2 (Matrix algebras)**.** Let $A = \mathrm{Mat}_n(K)$ be the algebra of $n \times n$ matrices over K and let $V = K^n$ be the K-vector space of column vectors with n rows.

We can transform any vector in V by multiplying it on the left by a matrix in A, and this makes V into an A-representation: in other words, given $X \in A$ and $v \in V$ let Xv just mean matrix multiplication.

This representation is irreducible since if $v, w \in W$ and $v \neq 0$ then some $X \in A$ has $Xv = w$. So every nonzero vector is *cyclic* in the sense that it is not contained in any proper A-subrepresentation.

In this case we have $\mathrm{End}(V) = A$, which is also an A-representation, via the *regular representation* in which one matrix acts on another by ordinary matrix multiplication $X : Y \mapsto XY$.

The regular representation of A is semsimple as we have $A \cong V^{\oplus n}$ as A-representations.

An explicit isomorphism $A \xrightarrow{\sim} V^{\oplus n}$ is the map sending

$$X = \begin{bmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{bmatrix} \mapsto \left(\begin{bmatrix} X_{11} \\ \vdots \\ X_{n1} \end{bmatrix}, \begin{bmatrix} X_{12} \\ \vdots \\ X_{n2} \end{bmatrix}, \cdots, \begin{bmatrix} X_{1n} \\ \vdots \\ X_{nn} \end{bmatrix} \right).$$

Notation. Here we define $V^{\oplus n}$ to be the set of n-tuples $(v_1, v_2, \dots, v_n)$ where each $v_i \in V$ and where

$$(v_1, v_2, \dots, v_n) + (w_1, w_2, \dots, w_n) \stackrel{\text{def}}{=} (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n),$$
$$c(v_1, v_2, \dots, v_n) \stackrel{\text{def}}{=} (cv_1, cv_2, \dots, cv_n),$$

for $v_i, w_i \in V$ and $c \in K$.

Example 2.3. More generally, suppose A is any algebra and V is an irreducible A-representation of finite dimension n. Then $\mathrm{End}(V) = \{\text{linear maps } L : V \to V\}$ is an A-representation for the action

$$a \cdot L : v \mapsto a \cdot L(v) \quad \text{for } a \in A \text{ and } v \in V.$$

This representation is semisimple with $\mathrm{End}(V) \cong V^{\oplus n}$ as A-representations. If V has basis $\{v_1, \cdots, v_n\}$ then an explicit isomorphism $\mathrm{End}(V) \xrightarrow{\sim} V^{\oplus n}$ is provided by the map $L \mapsto (L(v_1), \cdots, L(v_n))$

2.2 Subrepresentation of semisimple representations

Our main results today are derived from the following technical property. Among other consequences, it tells us that all subrepresentations of semisimple representations are semisimple.

Proposition 2.4. Let $V_1, V_2, \cdots, V_m$ be a finite list of irreducible finite-dimensional A-representations with $V_i \not\cong V_j$ if $i \neq j$. Consider the A-representation $V = \bigoplus_{i=1}^m V_i^{\oplus n_i}$ where $n_1, n_2, \ldots, n_m$ are nonnegative integers. Now suppose W is a subrepresentation of V. Then:

(1) For some integers $0 \leq r_i \leq n_i$ there is an isomorphism $\phi : \bigoplus_{i=1}^m V_i^{\oplus r_i} \xrightarrow{\sim} W$.

(2) The map $\bigoplus_{i=1}^m V_i^{\oplus r_i} \xrightarrow{\phi} W \hookrightarrow V$ is a direct sum of inclusions $\phi_i : V_i^{\oplus r_i} \hookrightarrow V_i^{\oplus n_i}$ of the form

$$\phi_i(a_1, a_2, \cdots, a_{r_i}) = \begin{bmatrix} a_1 & a_2 & \cdots & a_{r_i} \end{bmatrix} X_i$$

where each X_i is a full rank $r_i \times n_i$ matrix with values in K.

Proof sketch. If $W = 0$ then the proposition is trivial. Assume $W = 0$.

We proceed by induction on $n \overset{\text{def}}{=} n_1 + n_2 + \cdots + n_m$.

If $n = 1$ then we must have $0 \neq W = V$ in which case the result is again obvious.

Assume $n > 1$. Since W is finite-dimensional, it has an irreducible subrepresentation P (this was shown in the HW1 exercises). Observe that $\mathrm{Hom}_A(P, V) = \bigoplus_{i=1}^m \mathrm{Hom}_A(P, V_i)^{\oplus n_i}$. In this equation:

- each term $\mathrm{Hom}_A(P, V_i)$ on the right side is nonzero if and only if $P \cong V_i$ by Schur's lemma;
- the left side is nonzero since it contains inclusion $P \hookrightarrow W \hookrightarrow V$.

Therefore P must be isomorphic to V_i for some i.

The inclusion $V_i \xrightarrow{\sim} P \hookrightarrow V_i^{\oplus n_i} \hookrightarrow V$ must be given by a map of the form

$$v \mapsto (q_1 v, \cdots, q_{n_i} v)$$

for some scalars $q_i \in K$ that are not all zero. This is because composing this map with each projection

$$(a_1, \cdots, a_{n_i}) \mapsto a_j \in V_i$$

is a morphism of A-representations $V_i \to V_i$, which must be a scalar map by Schur's lemma.

Let $g \in \mathrm{GL}_{n_i}(K) = \{\text{invertible } n_i \times n_i \text{ matrices}\}$ act on $V_i^{\oplus n_i}$ on the right by the formula

$$g : (v_1, v_2, \cdots, v_{n_i}) \mapsto \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} g$$

while acting on $V_j^{\oplus n_j}$ for $i \neq j$ as the identity. This gives a right action of the general linear group on V.

We may choose $g \in \mathrm{GL}_{n_i}(K)$ such that

$$Pg = \{(0, 0, \cdots, 0, v) : v \in V_i\} \subset V_i^{\oplus n_i}.$$

Then $Wg = W' \oplus V_i$ where $V_i = Pg$ and W' is the kernel of projection $Wg \to Pg$, which satisfies

$$W' \subset V_1^{\oplus n_1} \oplus \cdots \oplus V_i^{\oplus(n_i-1)} \oplus \cdots \oplus V_m^{\oplus n_m}.$$

Now we apply the proposition to W' by induction, and multiply the resulting inclusion by g^{-1}. □

Corollary 2.5. Assume the following setup:

- V is an irreducible finite-dimensional representation of A.
- The elements $v_1, v_2, \ldots, v_n \in V$ are linearly independent.
- The elements $w_1, w_2, \ldots, w_n \in V$ are arbitrary.

Then there exists an element $a \in A$ such that $av_i = w_i$ for all $i = 1, 2, \ldots, n$.

Proof. Assume no such element exists. Then the image of A under the map

$$a \mapsto (av_1, \cdots, av_n)$$

is a proper subrepresentation of $V^{\oplus n}$, which we denote by W.

By Proposition 2.4 we know that $W \cong V^{\oplus m}$ for some $0 \le m < n$ and there exists an inclusion

$$\phi : V^{\oplus m} \xrightarrow{\sim} W \hookrightarrow V^{\oplus n}$$

of the form $\phi(a_1, a_2, \cdots, a_m) = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} X$ where X is a full rank $m \times n$ matrix.

Since $(v_1, v_2, \cdots, v_n) \in W$, we may choose $a_i \in V$ such that $\phi(a_1, a_2, \cdots, a_m) = (v_1, v_2, \cdots, v_n)$.

Also, since $m < n$, there is nonzero vector $\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \in K^n$ such that $X \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = 0$. But now

$$\sum_{i=1}^n q_i v_i = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} X \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = 0$$

which contradicts the linear independence of $v_1, \cdots, v_n$. □

Theorem 2.6 (Density theorem)**.** Let (V, ρ) be an irreducible, finite-dimensional A-representation. Then the map $\rho : A \to \mathrm{End}(V)$ is surjective. Moreover, if $(V, \rho) = (V_1, \rho_1) \oplus \cdots \oplus (V_r, \rho_r)$ where each (V_i, ρ_i) is an irreducible A-representation, then the map $\bigoplus_{i=1}^r \rho_i : A \to \bigoplus_{i=1}^r \mathrm{End}(V_i)$ is also surjective.

Proof. For the first claim, choose any $L \in \mathrm{End}(V)$ and suppose $v_1, v_2, \ldots, v_n$ is a basis of V. Set $w_i = L(v_i)$. By the previous corollary, some $a \in A$ has $\rho(a)v_i = w_i$ for all i which means that $\rho(a) = L$.

The second claim is nontrivial since direct sums of surjective maps are not necessarily surjective. For example, the direct sum of the identity map becomes $x \mapsto (x, x, \cdots, x)$ which is usually not surjective.

The surjective property that we wish to show will be a consequence of the second part of Proposition 2.4.

Let $Y = \bigoplus_{i=1}^r \mathrm{End}(V_i)$. This is a semisimple A-representation as $\mathrm{End}(V_i) \cong V_i^{\oplus d_i}$ where $d_i = \dim V_i$.

By Proposition 2.4, the subrepresentation

$$W \stackrel{\text{def}}{=} \left(\bigoplus_{i=1}^r \rho_i \right)(A) \subset Y$$

is isomorphic to $\bigoplus_{i=1}^r V_i^{\oplus m_i}$ for some integers $0 \le m_i \le d_i$, and there is an inclusion

$$\phi : \bigoplus_{i=1}^{r} V_i^{\oplus m_i} \xrightarrow{\sim} W \hookrightarrow Y$$

that is given by a direct sum of inclusions $\phi_i : V_i^{\oplus m_i} \hookrightarrow V_i^{\oplus d_i}$.

Since each ρ_i is surjective, the composition of this inclusion with the projection $Y \to \mathrm{End}(V_i)$ is surjective. Hence each ϕ_i is surjective and $m_i = d_i$. This shows that $\bigoplus_i \rho_i$ is surjective. □