This document is a **transcript** of the lecture, so is more like an abbreviated set of lecture slides than complete lecture notes. For the latter, **consult the textbook** listed on the course webpage.

1 Review from last time

In the last lecture we defined *semisimple representations*, which are representations isomorphic to direct sums of irreducible representations.

Notation. If $V_1, V_2, \dots, V_n$ are vector spaces, then we view elements of the direct sum $V_1 \oplus V_2 \oplus \cdots \oplus V_n$ either as tuples $(v_1, v_2, \dots, v_n)$ or as row vectors $\begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$ where each $v_i \in V_i$.

Assume A is an algebra defined over an algebraically closed field K.

We proved the following technical result last time:

Proposition 1.1. Suppose $V_1, V_2, \cdots, V_m$ are irreducible, pairwise non-isomorphic, finite-dimensional A-representations. Choose positive integers $n_1, n_2, \dots, n_m$ and define $V = \bigoplus_{i=1}^m V_i^{\oplus n_i}$. Then any sub-representation W of V has $W \cong \bigoplus_{i=1}^m V_i^{\oplus r_i}$ for some integers $0 \le r_i \le n_i$, and there is an isomorphism

$$\phi : \bigoplus_{i=1}^m V_i^{\oplus r_i} \xrightarrow{\sim} W$$

that sends $x = \begin{bmatrix} \underbrace{x_{11}\ x_{12}\ \cdots x_{1r_1}}_{\in V_1} & \underbrace{x_{21}\ x_{22}\ \cdots\ x_{2r_2}}_{\in V_2} & \cdots \end{bmatrix} \in \bigoplus_{i=1}^m V_i^{\oplus r_i}$ to $xM \in W$, where M is a full rank, block diagonal matrix with entries in K, whose successive blocks have size $r_i \times n_i$ for $i = 1, 2, \dots, m$.

Here are two consequences of this proposition:

- If V is an irreducible finite dimensional A-representation and $v_1, v_2, \dots, v_n \in V$ are linearly independent, then the map $a \mapsto (av_1, \dots, av_n)$ is a surjection $A \to V^{\oplus n}$.
- *Density theorem*: If $(V, \rho) = \bigoplus_{i=1}^r (V_i, \rho_i)$ is a direct sum of pairwise non-isomorphic, irreducible, finite dimensional A-representations, then $\bigoplus_{i=1}^r \rho_i : A \to \bigoplus_{i=1}^r \mathsf{End}(V_i)$ is surjective.

2 Matrix algebras

We have already seen that the algebra of all $n \times n$ matrices over K has a unique isomorphism class of irreducible representations. We can generalize this to block diagonal matrix algebras.

Choose integers $d_1, d_2, \dots, d_r > 0$.

Let $A = \bigoplus_{i=1}^r \mathsf{Mat}_{d_i}(K)$ where we define $\mathsf{Mat}_d(K)$ to be the algebra of $d \times d$ matrices over K.

Set $n = \sum_{i=1}^r d_i$. Then we can view A as the subalgebra of $\mathsf{Mat}_n(K)$ consisting of all block diagonal matrices with successive blocks of size $d_i \times d_i$.

The vector space K^n is automatically an A-representation. We construct a sequence of sub-representations:

Let $V_1 \subseteq K^n$ be the subspace of vectors with zeros outside rows $1, 2, \dots, d_1$

Let $V_2 \subseteq K^n$ be the subspace of vectors with zeros outside rows $d_1 + 1, d_1 + 2, \dots, d_1 + d_2$.

Let $V_3 \subseteq K^n$ be the subspace of vectors with zeros outside rows $d_1 + d_2 + 1, d_1 + d_2 + 2, \dots, d_1 + d_2 + d_3$.

Define $V_4, \dots, V_r$ analogously, so $V_r \subseteq K^n$ is the subspace of vectors with zeros outside the last d_r rows.

As vector spaces, we have $V_i \cong K^{d_i}$.

Theorem 2.1. In this setup, each V_i is an irreducible A-representation, and every finite-dimensional A-representation is isomorphic to a direct sum of zero or more copies of $V_1, V_2, \ldots, V_r$.

Before proving this theorem, we introduce another definition.

Definition 2.2. Suppose (V, ρ) is an A-representation.

Let V^* be the vector space of all K-linear maps $\lambda : V \to K$.

Then let $\rho^* : A \to \mathsf{End}(V^*)$ be the linear map defined by

$$\rho^*(a)(\lambda) : x \mapsto \lambda(\rho(a)(x)) \quad \text{for } a \in A \text{ and } \lambda \in V^*.$$

We refer to the pair (V^*, ρ^*) as the *dual* of (V, ρ).

It is a representation of the opposite algebra A^{op}.

Fact 2.3. For $A = \bigoplus_{i=1}^r \mathsf{Mat}_{d_i}(K) \subseteq \mathsf{Mat}_n(K)$, the usual matrix transpose map $X \mapsto X^\top$ is an algebra isomorphism $A \cong A^{\mathsf{op}}$.

Given a linear map between vector spaces $L : V \to W$, define $L^* : W^* \to V^*$ by $L^*(f) = f \circ L$.

Fact 2.4. If L is injective then L^* is surjective, and if L is surjective then L^* is injective.

Proof of Theorem 2.1. It is easy to see that each V_i is an irreducible A-representation, as each nonzero element of V_i is cyclic for the action of A.

Let X be some finite m-dimensional representation of A where $m < \infty$.

Then X^* is representation of $A^{\mathsf{op}} \cong A$.

In other words, X^* can be viewed as an A-representation for the action

$$a \cdot \lambda : x \mapsto \lambda(a^\top x) \quad \text{for } x \in X,\ \lambda \in X^*,\ a \in A.$$

Choose a basis $\{\lambda_1, \ldots, \lambda_m\}$ for X^*. Then let $\phi : A \oplus A \oplus \cdots \oplus A = A^{\oplus m} \to X^*$ be the map

$$\phi(a_1, a_2, \ldots, a_m) = a_1\lambda_1 + a_2\lambda_2 + \cdots + a_m\lambda_m.$$

Because $K \subset A$, this map is surjective. Therefore, the dual map $\phi^* : X \to (A^{\oplus m})^*$ is injective.

<u>Key claim</u>: The A-representations $(A^{\oplus m})^*$ and $A^{\oplus m}$ are isomorphic.

If we can prove this, then it will follow that X is isomorphic to a subrepresentation of $A^{\oplus m}$. As we have $A \cong \bigoplus_{i=1}^r V_i^{\oplus d_i}$ as A-representations (the isomorphism is provided by viewing a matrix as a tuple of column vectors), we would then get

$$X \cong \left(\text{a subrepresentation of } A^{\oplus m} \cong \bigoplus_{i=1}^r V_i^{\oplus m d_i} \right),$$

which by our technical proposition would imply that $X \cong \bigoplus_{i=1}^r V_i^{\oplus s_i}$ for some integers $s_i \geq 0$ as desired.

We will only explain the $m = 1$ case of the key claim.

Let A act on A^* by $a \cdot \lambda : x \mapsto \lambda(a^\top x)$ for $a \in A$ and $\lambda \in A^*$. Define $\Theta : A \to A^*$ to be the linear map

$$\Theta : a \mapsto (x \mapsto \mathrm{tr}(a^\top x)).$$

Then Θ is a bijection since it is a nonzero linear map with trivial kernel between finite-dimensional vector spaces of the same dimension. It is also a homomorphism of A-representations since we have

$$\Theta(gh)(x) = \mathrm{tr}(h^\top g^\top x) = \Theta(h)(g^\top x) = (g \cdot \Theta(h))(x) \quad \text{for } g, h, x \in A,$$

which implies that $\Theta(gh) = g \cdot \Theta(h)$. Thus $\Theta : A \xrightarrow{\sim} A^*$ is an isomorphism of A-representations. □

3 Filtrations

Continue to let A be an algebra. Suppose V is an A-representation.

Definition 3.1. A *filtration* of V is a finite, increasing sequence of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

where each V_i is sub-representation of V.

Lemma 3.2. If $\dim V < \infty$ then V has a filtration in which each quotient V_i/V_{i-1} is an irreducible A-representation.

Proof. We argue by induction on $\dim V$.

If $\dim V \leq 1$ then the result is trivial: just take $n = 1$ and $V_n = V$.

Assume $\dim V > 1$ and choose any irreducible subrepresentation $V_1 \subset V$.

Then let $U = V/V_1$. By induction there is a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} = U$$

in which each quotient U_i/U_{i-1} is irreducible.

Let V_i be the preimage of U_{i-1} under the quotient map $V \to V/V_1 = U$. Then

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

gives the desired filtration, since $V_i/V_{i-1} \cong (V_i/V_1)/(V_{i-1}/V_1) = U_{i-1}/U_{i-2}$ for $i > 1$. □

3.1 Radicals of finite-dimensional algebras

Assume that A is an algebra with $\dim A < \infty$.

Definition 3.3. The *radical* of A is the set of elements $a \in A$ that act as zero in every irreducible representation of A. Let $\mathsf{Rad}(A)$ denote this set of elements.

Proposition 3.4. The set $\mathsf{Rad}(A)$ is a two-sided ideal of A.

Proof. The set $\mathsf{Rad}(A)$ is a subspace of A since if (V, ρ) is a representation then

$$\rho(x) = 0 \implies \rho(cx) = c\rho(x) = 0 \quad \text{and} \quad \rho(x) = 0 = \rho(y) \implies \rho(x+y) = \rho(x) + \rho(y) = 0$$

for all $x, y \in A$ and $c \in K$. It is also a two-sided ideal since if $a, b \in A$ then

$$\rho(x) = 0 \implies \rho(axb) = \rho(a)\rho(x)\rho(b) = 0.$$

□

Let I be a two-sided idea in A. For integers $n \geq 1$, let $I^n = K\text{-span}\{x_1 x_2 \cdots x_n : x_i \in I\}$.

We say that I is *nilpotent* if $I^n = 0$ for some $n > 0$.

For example, the subspace of strictly upper triangular matrices is a nilpotent ideal in the algebra of all upper triangular $n \times n$ matrices over K.

Proposition 3.5. If I is a nilpotent two-sided ideal in A then $I \subseteq \mathsf{Rad}(A)$.

Proof. Suppose I is a nilpotent two-sided ideal with $I^n = 0$. Choose any irreducible A-representation V and pick $0 \leq v \in V$. Then the subspace $Iv \overset{\text{def}}{=} \{xv : x \in I\}$ is a subrepresentation. If $Iv = V$ then there is some $x \in I$ with $xv = v$, which is impossible as $x^n = 0$. Therefore $Iv = 0$ as it is a proper subrepresntation of an irreducible representation. Since V was arbitrary, it follows that $I \subseteq \mathsf{Rad}(A)$. □

The following shows that $\mathsf{Rad}(A)$ is precisely the largest nilpotent two-sided ideal in A.

Proposition 3.6. $\mathsf{Rad}(A)$ is a nilpotent two-sided ideal.

Proof. Since $\dim A < \infty$, the previous section shows that there exists a filtration of the regular representation $0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$ in which each quotient A_i/A_{i-1} is irreducible as an A-representation.

Each $x \in \mathsf{Rad}(A)$ acts as zero on A_i/A_{i-1}, which means that $xA_i \subset A_{i-1}$.

Therefore if $x_1, x_2, \cdots \in \mathsf{Rad}(A)$ then $x_1 x_2 \cdots x_i A \subset A_{n-i}$ and $x_1 x_2 \cdots x_n A = 0$. Hence $\mathsf{Rad}(A)^n = 0$. □

4 Representations of finite-dimensional algebras

As a final application today, we can "classify" all representations of finite-dimensional algebras.

Theorem 4.1. Suppose A is a finite-dimensional algebra. Then A has finitely many isomorphism classes of irreducible representations $V_1, V_2, \ldots, V_r$ and $A/\mathsf{Rad}(A) \cong \bigoplus_{i=1}^r \mathsf{End}(V_i)$ as K-algebras. Moreover, every irreducible A-representation is finite-dimensional.

Notice that since $\dim V_i$ is finite, we have $\mathsf{End}(V_i) \cong \mathsf{Mat}_d(K)$ for $d = \dim V_i$.

Therefore $A/\mathsf{Rad}(A)$ is isomorphic to a block diagonal matrix algebra of the form considered earlier today.

Proof. Suppose V is an A-representation.

If $0 \neq x \in V$ then Ax is a nonzero subrepresentation of dimension at most $\dim A < \infty$.

Therefore, if V is irreducible then we must have $V = Ax$ and $\dim V \leq \dim A < \infty$.

Now suppose $(V_1, \rho_1), \ldots, (V_r, \rho_r)$ are pairwise non-isomorphic, irreducible A-representations.

By the *density theorem*, the direct sum

$$\phi = \bigoplus_{i=1}^r \rho_i : A \to \bigoplus_{i=1}^r \mathsf{End}(V_i)$$

is a surjective map. Since each $\mathsf{End}(V_i)$ has dimension $(\dim V_i)^2$, we have

$$r \leq \sum_{i=1}^r (\dim V_i)^2 \leq \dim A < \infty$$

Thus r cannot be arbitrarily large, so the number of distinct isomorphism classes of irreducible A-representations is finite and at most $\dim A$.

Finally assume r is maximal above, so that every irreducible A-representation is isomorphic to some V_i. Then $\mathsf{Rad}(A) = \ker(\phi)$ so ϕ passes to an isomorphism $A/\mathsf{Rad}(A) \cong \bigoplus_{i=1}^{r} \mathsf{End}(V_i)$. $\square$

Corollary 4.2. If $V_1, V_2, \ldots, V_r$ are pairwise non-isomorphic irreducible representations of a finite-dimensional algebra A then $\sum_{i=1}^{r} (\dim V_i)^2 \leq \dim A$.