This document is a **transcript** of the lecture, so is more like an abbreviated set of lecture slides than complete lecture notes. For the latter, **consult the textbook** listed on the course webpage.

1 Review from last time

Let $\mathbb{K}$ be an algebraically closed field.

For integers $n > 0$, let $\mathrm{Mat}_n(\mathbb{K})$ be the algebra of $n \times n$ matrices over $\mathbb{K}$.

Remember that if V is an n-dimensional $\mathbb{K}$-vector space, then $\mathrm{End}(V) \cong \mathrm{Mat}_n(\mathbb{K})$.

Theorem 1.1. Suppose $A = \bigoplus_{i=1}^r \mathrm{Mat}_{d_i}(\mathbb{K})$ for some integers $d_1, d_2, \dots, d_r > 0$. For each index $i \in \{1, 2, \dots, r\}$, A has an irreducible representation V_i of dimension d_i, and every finite-dimensional representation of A is a direct sum of copies of $V_1, V_2, \dots, V_r$, which are pairwise non-isomorphic.

If we view $A \subseteq \mathrm{Mat}_n(\mathbb{K})$ as a subalgebra of block diagonal $n \times n$ matrices where $n = d_1 + d_2 + \cdots + d_r$, then we can construct V_i as the subspace of vectors in $\mathbb{K}^n$ with zeros outside the rows indexed by

$$(d_1 + d_2 + \cdots + d_{i-1}) + \{1, 2, \dots, d_i\}.$$

Lemma 1.2. Suppose A is any K-algebra with a finite-dimensional representation V. There exists a finite filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ where each V_i is a subrepresentation with V_i/V_{i-1} irreducible.

Let A be any $\mathbb{K}$-algebra with $\dim(A) < \infty$. The radical of A is

$$\begin{aligned} \mathrm{Rad}(A) &\overset{\text{def}}{=} \{\text{the elements in } A \text{ that act as zero in every irreducible representation of } A\} \\ &= (\text{the largest nilpotent 2-sided ideal in } A). \end{aligned}$$

Theorem 1.3. A finite-dimensional algebra A has finitely many irreducible representations $V_1, \dots, V_r$ up to isomorphism, each representation V_i has finite dimension $d_i = \dim(V_i)$, and it holds that

$$A/\mathrm{Rad}(A) \cong \bigoplus_{i=1}^r \mathrm{End}(V_i) \cong \bigoplus_{i=1}^r \mathrm{Mat}_{d_i}(\mathbb{K}).$$

Since each $\mathrm{End}(V_i)$ has dimension $d_i^2 = \dim(V_i)^2$, it follows that:

Corollary 1.4. If $\dim A < \infty$, then $\dim A - \dim \mathrm{Rad}(A) = \sum_{i=1}^r \dim(V_i)^2 \leq \dim A$.

Example 1.5. Suppose $A = \mathbb{K}[x]/(x^n)$ where $n \geq 1$. Then we can view A as the vector space

$$A = \mathbb{K}\text{-span}\{1, x, \dots, x^{n-1}\}.$$

The fact that

$$\begin{aligned} x^n = 0 \text{ in } A &\implies \text{if } (\rho, V) \text{ is a finite-dimensional representation of } A, \text{ then exists a basis for } V \\ &\qquad \text{in which matrix of } \rho(x) \text{ is strictly upper triangular} \\ &\implies \text{if } V \text{ is irreducible, then } \rho(x) = 0 \text{ and } \dim V = 1. \end{aligned}$$

Thus $A/\mathrm{Rad}(A) \cong \mathrm{End}(\mathbb{K}) = \mathbb{K}$.

Note that we can see directly that $\mathrm{Rad}(A) = (x)$ as this is the largest nilpotent two-sided ideal in A.

Example 1.6. Suppose A is the subalgebra of upper-triangular matrices in $\mathrm{Mat}_n(\mathbb{K})$.

Let (V_i, ρ_i) be the representation of A in which $V_i = \mathbb{K}$ and $\rho_i : A \to \mathrm{End}(\mathbb{K}) = \mathbb{K}$ sends $a \in A$ to the scalar $\rho_i(a) = a_{ii}$ (the diagonal entry of a in row i) for $i = 1, 2, \dots, n$.

One can check that representations are irreducible and pairwise-non-isomorphic.

They give all irreducible representations of A (up to isomorphism) since

$$\mathrm{Rad}(A) = \{\text{strictly upper-triangular matrices in } \mathrm{Mat}_n(\mathbb{K})\},$$

as this is the largest nilpotent two-sided ideal in A, and so

$$A/\mathrm{Rad}(A) \cong \mathbb{K}^n \implies \text{there are exactly } n \text{ isomorphism classes of irreducible } A\text{-representations.}$$

2 Semisimple algebras

Our main new results today concern the following class of algebras.

Definition 2.1. A finite-dimensional algebra A is called *semisimple* if $\mathrm{Rad}(A) = 0$.

Recall that a representation is *semisimple* if it is a direct sum of irreducible representations.

Proposition 2.2. Assume A is an algebra over $\mathbb{K}$ with $\dim A < \infty$. The following are equivalent:

(1) A is semisimple.

(2) $\sum_{i=1}^{r} \dim(V_i)^2 = \dim A$ where V_i are the distinct isomorphism classes of irreducible A-representations.

(3) $A \cong \bigoplus_{i=1}^{r} \mathrm{Mat}_{d_i}(\mathbb{K})$ for some integers $d_1, d_2, \dots, d_r > 0$

(4) Any finite-dimensional representation of A is semisimple.

(5) The regular representation of A is semisimple.

Proof. We have (1) $\iff$ (2) since $\dim A - \dim \mathrm{Rad}(A) = \sum_{i=1}^{r} \dim(V_i)^2$.

The implication (1) $\implies$ (3) is Theorem 1.3. Conversely, (3) + Theorem 1.1 $\implies$ (2) $\implies$ (1).

We conclude that (1) $\iff$ (3).

Now we claim that (3) $\implies$ (4) $\implies$ (5) $\implies$ (3).

The implication (3) $\implies$ (4) holds by Theorem 1.1 and (4) $\implies$ (5) is trivial.

To show that (5) $\implies$ (3), assume (5). Then we can write $A = \bigoplus_{i=1}^{r} d_i V_i$ where $V_1, V_2, \dots, V_r$ are irreducible and pairwise-non-isomorphic, since the regular representation of A is semisimple.

Now consider $\mathrm{End}_A(A) = \{\text{morphisms } A \to A \text{ as } A\text{-representations}\} = \mathrm{Hom}_A(A, A)$.

Schur's lemma tells us that

- $\mathrm{End}_A(V_i) = \mathbb{K}$ so $\mathrm{End}_A(d_i V_i) \cong \mathrm{Mat}_{d_i}(\mathbb{K})$, and
- $\mathrm{Hom}_A(V_i, V_j) = 0$ if $i \neq j$, so $\mathrm{Hom}_A(d_i V_i, d_j V_j) = 0$ if $i \neq j$.

Thus, we compute $\mathrm{End}_A(A) = \mathrm{Hom}_A(A, A) = \bigoplus_{i,j} \mathrm{Hom}(d_i V_i, d_j V_j) \cong \bigoplus_i \mathrm{Mat}_{d_i}(\mathbb{K})$.

Exercise: Show that $(\mathrm{End}_A(A))^{\mathsf{op}} \cong A$ or equivalently that $\mathrm{End}_A(A) \cong A^{\mathsf{op}}$.

Last time: There is an isomorphism $(\bigoplus_i \mathrm{Mat}_{d_i}(\mathbb{K}))^{\mathsf{op}} \cong \bigoplus_i \mathrm{Mat}_{d_i}(\mathbb{K})$ afforded by the transpose map.

Thus we have $A \cong (\mathrm{End}_A(A))^{\mathsf{op}} \cong (\bigoplus_i \mathrm{Mat}_{d_i}(\mathbb{K}))^{\mathsf{op}} \cong \bigoplus_i \mathrm{Mat}_{d_i}(\mathbb{K})$.

This is property (3), so (5) $\implies$ (3) as desired. □

3 Characters

Let A be an algebra. Suppose (V, ρ) is a finite-dimensional representation of A.

Definition 3.1. The *character* of (V, ρ) is the linear map $\chi_{(V,\rho)} : A \to \mathbb{K}$ with the formula

$$\chi_{(V,\rho)}(a) = \mathrm{tr}(\rho(a)) \quad \text{for } a \in A.$$

How can we compute the trace of $\phi \in \mathrm{End}(V)$?

First choose a basis $e_1, e_2, \dots, e_n$ of V. Then $\mathrm{tr}(\phi) = \sum_{i=1}^{n}(\text{coefficient of } e_i \text{ in } \phi(e_i))$.

Some basic and well-known facts about traces:

(1) The method just given to compute the trace does not depend on the choice of basis.

(2) We have $\mathrm{tr}(\phi_1\phi_2) = \mathrm{tr}(\phi_2\phi_1)$ for all $\phi_1, \phi_2 \in \mathrm{End}(V)$, so $\mathrm{tr}(\phi_1\phi_2\phi_1^{-1}) = \mathrm{tr}(\phi_2)$ if ϕ_1 is invertible.

(3) If $(V_1, \rho_1) \cong (V_2, \rho_2)$ are finite-dimensional A-representations then $\chi_{(V_1,\rho_1)} = \chi_{(V_2,\rho_2)}$.

To abbreviate, we will sometimes write χ_V instead of $\chi_{(V,\rho)}$.

Let $[A, A] = \mathbb{K}\text{-span}\left\{[a, b] \overset{\text{def}}{=} ab - ba : a, b \in A\right\}$. We view this as just a vector space.

Fact 3.2. We always have $[A, A] \subseteq \ker(\chi_{(V,\rho)})$

Proof. Let $\chi = \chi_{(V,\rho)}$. Then $\chi(ab - ba) = \mathrm{tr}(\rho(ab)) - \mathrm{tr}(\rho(ba)) = \mathrm{tr}(\rho(a)\rho(b)) - \mathrm{tr}(\rho(b)\rho(a)) = 0$ □

In the following theorem, $\dim A$ is not required to be finite.

Theorem 3.3. The characters of any list of non-isomorphic irreducible finite-dimensional A-representations are linearly independent (and, in particular, are distinct).

Proof. Suppose $(V_1, \rho_1), (V_2, \rho_2), \dots, (V_r, \rho_r)$ are pairwise non-isomorphic irreducible finite-dimensional A-representations. Let $\chi_i = \chi_{(V_i,\rho_i)}$. By the density theorem, the map

$$\rho_1 \oplus \cdots \oplus \rho_r : A \to \mathrm{End}(V_1) \oplus \cdots \oplus \mathrm{End}(V_r)$$

is surjective. Therefore, if $\sum_{i=1}^{r} \lambda_i\chi_i(a) = 0$ for all $a \in A$ for some coefficients $\lambda_1, \lambda_2, \dots, \lambda_r \in \mathbb{K}$, then

$$\sum_{i=1}^{r} \lambda_i \mathrm{tr}(M_i) = 0 \quad \text{for any } M_i \in \mathrm{End}(V_i) \text{ chosen independently,}$$

which is only possible if $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$. □

We say that a character $\chi_{(V,\rho)}$ is *irreducible* if (V, ρ) is irreducible.

Theorem 3.4. Assume A is semisimple and $\dim A < \infty$. Then the irreducible characters of A are a basis for the vector space $(A/[A, A])^*$ of linear maps $A/[A, A] \to \mathbb{K}$.

Proof. Each character χ has $[A, A] \subset \ker(\chi)$, so χ belongs to $(A/[A, A])^*$.

Since $A = \mathrm{Mat}_{d_1}(\mathbb{K}) \oplus \cdots \oplus \mathrm{Mat}_{d_r}(\mathbb{K})$ it follows that $[A, A] = \bigoplus_{i=1}^{r}[\mathrm{Mat}_{d_i}(\mathbb{K}), \mathrm{Mat}_{d_i}(\mathbb{K})]$.

We claim that $[\mathrm{Mat}_d(\mathbb{K}), \mathrm{Mat}_d(\mathbb{K})] = \mathfrak{sl}_d(\mathbb{K})$, where $\mathfrak{sl}_d(\mathbb{K})$ is the vector space of $d \times d$ matrices over $\mathbb{K}$ with zero trace. To prove the claim, note that the trace map certainly vanishes on $[\mathrm{Mat}_d(\mathbb{K}), \mathrm{Mat}_d(\mathbb{K})]$ and that $\mathfrak{sl}_d(\mathbb{K})$ is spanned by the commutators

$$E_{ij} = [E_{ik}, E_{kj}] \text{ for } i \neq j \quad \text{and} \quad E_{ii} - E_{i+1,i+1} = [E_{i,i+1}, E_{i+1,i}]$$

where E_{ij} is the elementary matrix with 1 in entry (i, j) and 0 elsewhere.

With the claim proved, we have $A/[A, A] \cong \mathbb{K}^r$ since $\mathrm{Mat}_d(\mathbb{K})/\mathfrak{sl}_d(\mathbb{K}) \cong \mathbb{K}$.

Finally, we know that A has r distinct irreducible characters (by Theorem 1.1), and these are linearly independent elements of $(A/[A, A])^*$, so they must be a basis as $\dim(A/[A, A])^* = \dim(A/[A, A]) = r$. □

4 Two general results

We finish today with two general results that can be applied to algebras A that are not necessarily semisimple. Assume $\dim A < \infty$. Let V be a finite-dimensional representation of A.

Theorem 4.1. (*Jordan-Hölder theorem*) Suppose we have filtrations

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V \quad \text{and} \quad 0 = V_0' \subset V_1' \subset \cdots \subset V_m' = V$$

where V_i and V_i' are subrepresentations such that the quotients $W_i \overset{\text{def}}{=} V_i/V_{i-1}$ and $W_i' \overset{\text{def}}{=} V_i'/V_{i-1}'$ are irreducible. Then $n = m$ and there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $W_{\sigma(i)} \cong W_i'$ for all i.

We call the common length $m = n$ of these filtrations the *length* of the representation V.

Proof. We can give a simple proof when $\mathrm{char}(\mathbb{K}) = 0$. In this case, it follows by a homework exercise that $\chi_V = \chi_W + \chi_{V/W}$ if A is any subrepresentation of V, and so we have $\chi_V = \sum_{i=1}^n \chi_{W_i} = \sum_{i=1}^m \chi_{W_i'}$.

Then we can deduce the theorem by the linear independence of the irreducible characters of A.

This argument does not work for $\mathrm{char}(\mathbb{K}) = p > 0$, because the multiplicities of the irreducible characters in the decomposition of χ_V could be multiples of p. One can handle this case by a more involved inductive argument; see the textbook for the details. □

We maintain the same setup for A and V in the next theorem.

Theorem 4.2. (*Krull-Schmidt theorem*) There is a decomposition of V, which is unique up to isomorphism and rearrangement of factors, as a direct sum of indecomposable A-representations.

We will give the proof next time. While the existence of such a decomposition follows pretty easily by induction on $\dim V$, the uniqueness claim in the theorem is nontrivial.