This document is a **transcript** of the lecture, so is more like an abbreviated set of lecture slides than complete lecture notes. For the latter, **consult the textbook** listed on the course webpage.

1 Review from last time

1.1 Two structure theorems

We saw two main theorems in the previous lecture, concerning finite-dimensional representations of an arbitrary algebra A over an algebraically closed field $\mathbb{K}$.

Theorem 1.1 (*Jordan-Hölder theorem*)**.** Any finite-dimensional A-representation V has a filtration with irreducible quotients, and the length of the filtration and the isomorphism classes of the quotients are uniquely determined up to permutation of indices.

Theorem 1.2 (*Krull-Schmidt theorem*)**.** Any finite-dimensional A-representation V has a direct sum decomposition into indecomposable subrepresentations that is unique up to isomorphism and rearrangement of factors.

Remark 1.3. In both theorems, the algebra A can have infinite dimension as long as $\dim V < \infty$, since the relevant statements hold for V viewed as an A-representation if and only if they hold for V viewed as a representation of the finite-dimensional algebra $\rho_V(A) \subseteq \mathrm{End}(V)$.

More strongly, one can show that the results even hold when the field $\mathbb{K}$ is not algebraically closed, though one might need to revise our proofs from last time.

1.2 Representations of tensor products

We also discussed the representations of the tensor product of two algebras. Let A and B be $\mathbb{K}$-algebras. Then the vector space $A \otimes B$ is also a $\mathbb{K}$-algebra for the product defined on pure tensors by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) \stackrel{\text{def}}{=} a_1a_2 \otimes b_1b_2 \quad \text{for } a_1, a_2 \in A,\ b_1, b_2 \in B.$$

If V is an A-representation and W is a B-representation then $V \otimes W$ is an $A \otimes B$-representation with

$$(a \otimes b) \cdot (v \otimes w) \stackrel{\text{def}}{=} av \otimes bw \quad \text{for } a \in A,\ b \in B,\ v \in V,\ w \in W.$$

Theorem 1.4. Suppose V is an A-representation and W is a B-representation. If V and W are both irreducible and finite-dimensional then so is $V \otimes W$ (viewed as an $A \otimes B$-representation). Moreover, up to isomorphism, all irreducible finite-dimensional representations of $A \otimes B$ arise in this way.

We reduced the proof of this theorem to the case when A and B are semisimple and finite-dimensional, in which case each algebra is isomorphic to a direct sum of matrix algebras.

In this concrete setting, the theorem is easy to check directly. For example, $\mathrm{Mat}_m(\mathbb{K})$ and $\mathrm{Mat}_n(\mathbb{K})$ have unique irreducible representations $\mathbb{K}^m$ and $\mathbb{K}^n$ up to isomorphism, and $\mathbb{K}^m \otimes \mathbb{K}^n \cong \mathbb{K}^{mn}$ is the unique irreducible representation of $\mathrm{Mat}_{mn}(\mathbb{K}) \cong \mathrm{Mat}_m(\mathbb{K}) \otimes \mathrm{Mat}_n(\mathbb{K})$ up to isomorphism.

2 Finite group representations

For a $\mathbb{K}$-vector space V recall that $\mathrm{GL}(V)$ is the group of invertible linear maps $V \to V$.

Let G be a group. Recall that a *(group) representation* of G is a pair (V, ρ) where V is a vector space and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism. In earlier lectures, we saw that representations of G are the same thing as representations of the group algebra $\mathbb{K}[G] = \mathbb{K}\text{-span}\{a_g : g \in G\}$.

From now on, we will think of elements of $\mathbb{K}[G]$ as formal (finite) linear combinations of group elements, writing $\sum_{g\in G} c_g g$ instead of $\sum_{g\in G} c_g a_g$, where $c_g \in \mathbb{K}$ and a_g is the formal symbol indexed by $g \in G$.

We are interested in representations of **finite** groups G. In this case $\mathbb{K}[G]$ has finite dimension.

Our first important question to answer is: when is $\mathbb{K}[G]$ semisimple?

From this point on, assume that the group G is finite. Write $|G|$ for its number of elements.

Theorem 2.1 (*Maschke's theorem*)**.** Assume that $\mathrm{char}(\mathbb{K})$ does not divide $|G|$. Then $\mathbb{K}[G]$ is semisimple.

Proof. Let (V, ρ) be a finite-dimensional G-representation, and hence also a $\mathbb{K}[G]$-representation.

It suffices to check that V is a direct sum of irreducible subrepresentations. This clearly holds if (V, ρ) is irreducible so assume this is not the case. Then V must have an irreducible subrepresentation W by one of our homework exercises. By induction on dimension, it is enough to show that V has another nonzero subrepresentation U such that $V = W \oplus U$.

We can find a subspace $\widetilde{U}$, not necessarily a subrepresentation, with $V = W \oplus \widetilde{U}$ as vector spaces.

Just choose a basis $w_1, w_2, \dots, w_m$ of W, extend this to a basis $w_1, \dots, w_m, u_1, \dots, u_n$ for V, and set

$$\widetilde{U} = \mathbb{K}\text{-span}\{u_1, \dots, u_n\}.$$

Here is the key idea to the proof.

Let $\pi : V \to W$ be linear map with $\pi(w_i) = w_i$ for all i and $\pi(u_j) = 0$ for all j. Then define

$$\sigma = \frac{1}{|G|} \sum_{g\in G} \rho(g) \circ \pi \circ \rho(g^{-1})$$

Finally consider $U = \mathsf{kernel}(\sigma)$. We claim that:

(1) U is a subrepresentation.

(2) $V = W \oplus U$.

Property (1) holds because for any $h \in G$ we have

$$\sigma\rho(h) = \frac{1}{|G|} \sum_{g\in G} \rho(g) \circ \pi \circ \rho(g^{-1}h) = \frac{1}{|G|} \sum_{x\in G} \rho(hx) \circ \pi \circ \rho(x^{-1}) = \rho(h)\sigma,$$

making the substitution $x = h^{-1}g$ in the second equality.

Thus $\sigma(u) = 0$ if and only if $\sigma\rho(h)(u) = \rho(h)\sigma(u) = 0$ for any $h \in G$ and $u \in U$, as $\rho(h)$ is invertible.

For property (2), note that $\rho(W) \subseteq W$ and $\pi(w) = w$ for all $w \in W$, so $\sigma(w) = w$ for all $w \in W$.

Since $\sigma(V) \subseteq W$, it follows that $\sigma^2 = \sigma$. Thus any $v \in V$ can be written as

$$v = \sigma(v) + (v - \sigma(v))$$

where $\sigma(v) \in W$ and $(v - \sigma(v)) \in U$, and we have $W \cap U = 0$ since if $x \in W \cap U$ then $x = \sigma(x) = 0$.

Thus $V = W \oplus U$ as needed. $\square$

Corollary 2.2. Assume $\mathrm{char}(\mathbb{K})$ does not divide $|G|$. Then there are finitely many isomorphism classes of irreducible G-representations $\{(V_i, \rho_i)\}_{i\in I}$, all of which have finite dimension, and we have

$$|G| = \textstyle\sum_{i\in I} (\dim V_i)^2 \quad \text{and} \quad \mathbb{K}[G] \cong \bigoplus_{i\in I} \mathrm{End}(V_i).$$

The representation theory of finite-dimensional semisimple algebras is trivial in the sense that everything is just a direct sum of matrix algebras. What makes the representation theory of finite groups more interesting is the distinguished basis of $\mathbb{K}[G]$ provided by G itself. Going from this basis to the natural bases of $\mathbb{K}[G]$ viewed as a sum of matrix algebras is nontrivial.

It turns out that the converse to Maschke's theorem is also true.

Theorem 2.3 (Converse to Maschke's theorem)**.** If $\mathbb{K}[G]$ is semisimple then $\mathrm{char}(\mathbb{K})$ does not divide $|G|$.

Proof. Assume $\mathbb{K}[G]$ is semisimple and consider the subspace

$$U \overset{\text{def}}{=} \mathbb{K}\text{-span}\left\{\textstyle\sum_{g\in G} g\right\}.$$

This is a 1-dimensional subrepresentation of $\mathbb{K}[G]$.

By semisimplicity, there exists a complementary subrepresentation $V \subset \mathbb{K}[G]$ with $\mathbb{K}[G] = U \oplus V$.

View $\mathbb{K}$ as a G-representation with $g \cdot c = c$ for all $g \in G$ and $c \in \mathbb{K}$.

Then define $\phi : \mathbb{K}[G] \to \mathbb{K}$ to be linear map that sends $V \to 0$ and $\sum_{g\in G} g \mapsto 1_{\mathbb{K}}$.

Because U and V are subrepresentations, the map ϕ is a morphism of $\mathbb{K}[G]$-representations.

Thus $\phi(g) = \phi(g \cdot 1_G) = g \cdot \phi(1_G) = \phi(1_G) \in \mathbb{K}$ for all $g \in G$.

But this means that

$$1_{\mathbb{K}} = \phi(\textstyle\sum_{g\in G} g) = \sum_{g\in G} \phi(g) = \sum_{g\in G} \phi(1_G) = |G|\phi(1_G).$$

Thus $|G|$ is invertible (and nonzero) in $\mathbb{K}$, so $\mathrm{char}(\mathbb{K})$ must not divide $|G|$. □

3 Characters of group representations

Continue to let G be a finite group.

If (V, ρ) is a G-representation with $\dim V < \infty$ then its *character* is the map $\chi_{(V,\rho)} : G \to \mathbb{K}$ with

$$\chi_{(V,\rho)}(g) = \mathrm{tr}(\rho(g)).$$

Since traces are invariant under change of basis, it follows that:

Fact 3.1. If $(V, \rho) \cong (V', \rho')$ as G-representations then $\chi_{(V,\rho)} = \chi_{(V',\rho')}$.

The *conjugacy classes* of G are the sets $\mathcal{K}_g \overset{\text{def}}{=} \{xgx^{-1} : x \in G\}$ for $g \in G$.

A *class function* of G is a map $G \to \mathbb{K}$ that is constant on all elements in each conjugacy class.

Equivalently, $f : G \to \mathbb{K}$ is a class function if and only if $f(xgx^{-1}) = f(g)$ for all $x, g \in G$.

Fact 3.2. The character of any finite-dimensional G-representation is a class function.

We say that the character $\chi_{(V,\rho)}$ is *irreducible* if (V, ρ) is an irreducible representation.

We mention some special properties of irreducible characters that hold when $\mathbb{K}[G]$ is semisimple.

Proposition 3.3. If $\mathrm{char}(\mathbb{K})$ does not divide $|G|$ then the irreducible characters of G are a basis for the vector space of class functions of G.

Proof. In this case $\mathbb{K}[G]$ is semisimple so the irreducible characters are a basis for $(\mathbb{K}[G]/[\mathbb{K}[G],\mathbb{K}[G]])^*$.

By definition, this dual space can be identified with the vector space of linear maps $f : G \to \mathbb{K}$ that satisfy $f(XY) = f(YX)$ for all $X, Y \in \mathbb{K}[G]$. Check that this is the same as the set of linear maps $f : G \to \mathbb{K}$ with $f(gh) = f(hg)$ for all $g, h \in G$, or equivalently with $f(xgx^{-1}) = f(g)$ for all $x, g \in G$.

Thus, we can identify $(\mathbb{K}[G]/[\mathbb{K}[G],\mathbb{K}[G]])^*$ with the vector space of class functions of G. □

Corollary 3.4. If $|G|$ is not divisible by $\mathrm{char}(\mathbb{K})$ then the number of isomorphism classes of irreducible G-representations is the same as the number of distinct irreducible characters of G, which is also the number of distinct conjugacy classes of G.

Corollary 3.5. If $\mathrm{char}(\mathbb{K}) = 0$ then two finite-dimensional G-representations are isomorphic if and only if they have the same character.

A group G is *abelian* if $gh = hg$ for all $g, h \in G$.

This holds if and only if the group algebra $\mathbb{K}[G]$ is commutative, so the following is true:

Fact 3.6. If G is abelian then all irreducible G-representations are 1-dimensional.

Suppose $f : V \to W$ is a linear map between vector spaces.

Recall that V^* is the vector space of linear maps $\lambda : V \to \mathbb{K}$.

Define $f^* : W^* \to V^*$ to be the linear map with the formula $f^*(\lambda) = \lambda \circ f$.

If $f \in \mathrm{GL}(V)$ then $f^* \in \mathrm{GL}(V^*)$ since $(f \circ g)^* = g^* \circ f^*$.

Now suppose (V, ρ_V) is a G-representation. Define $\rho_{V^*} : G \to \mathrm{GL}(V^*)$ by the formula

$$\rho_{V^*}(g) = (\rho_V(g)^*)^{-1} = (\rho_V(g)^{-1})^* = \rho_V(g^{-1})^*.$$

Fact 3.7. If (V, ρ_V) is a representation then so is (V^*, ρ_{V^*}).

From this point on, we assume $\dim V < \infty$.

Fact 3.8. We have $\mathrm{tr}(f) = \mathrm{tr}(f^*)$ so $\chi_{(V^*,\rho^*)}(g) = \chi_{(V,\rho)}(g^{-1})$ for all $g \in G$.

Since G is a finite group, any $g \in G$ has $g^{|G|} = 1_G$, and so any eigenvalue of $\rho_V(g)$ is a root of unity.

The character value $\chi_{(V,\rho_V)}(g)$ is the sum of the eigenvalues of $\rho_V(g)$, and is therefore a sum of roots of unity in $\mathbb{K}$. When $\mathbb{K} = \mathbb{C}$, the inverse of any root of unity is its complex conjugate.

As the eigenvalues of $\rho_V(g^{-1})$ are the inverses of the eigenvalues of $\rho_V(g)$, we deduce that:

Fact 3.9. If $\mathbb{K} = \mathbb{C}$ then $\overline{\chi_{(V,\rho_V)}(g)} = \chi_{(V,\rho_V)}(g^{-1}) = \chi_{(V^*,\rho_{V^*})}(g)$ for all $g \in G$.

In this case $(V, \rho_V) \cong (V^*, \rho_{V^*})$ if and only if $\chi_{(V,\rho_V)}$ takes all **real** values.

Finally suppose (V, ρ_V) and (W, ρ_W) are G-representations. Then $(V \otimes W, \rho_{V\otimes W})$ is a G-representation when $\rho_{V\otimes W}(g)$ is linear map sending $v \otimes w \mapsto \rho_V(g)(v) \otimes \rho_W(g)(w)$ for $g \in G$, $v \in V$, and $w \in W$.

Fact 3.10. If $\dim V < \infty$ and $\dim W < \infty$ then $\chi_{(V\otimes W,\rho_{V\otimes W})} = \chi_{(V,\rho_V)}\chi_{(W,\rho_W)}$.