This document is a **transcript** of the lecture, so is more like an abbreviated set of lecture slides than complete lecture notes. For the latter, **consult the textbook** listed on the course webpage.

1 Review from last time

Let K be an algebraically closed field and let G be a group.

A *representation* of G is a representation (V, ρ) of the group algebra $K[G]$.

Equivalently, a G-representation is a pair (V, ρ) such that $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism.

In this case $\rho(K[G]) \subseteq \mathrm{End}(V)$ and $\rho(G) \subseteq \mathrm{GL}(V)$.

Assume G is a finite group.

Theorem 1.1 (*Maschke's theorem*)**.** The group algebra $K[G]$ is *semisimple* (meaning that all irreducible G-representations are finite-dimensional and finite-dimensional G-representations are direct sums of irreducible representations) if and only if $\mathrm{char}(K)$ does not divide $|G|$.

Assume (V, ρ) is a finite dimensional G-representation.

The *character* of (V, ρ) is a linear map $\chi_{(V,\rho)} : K[G] \to K$ sending $g \mapsto \mathrm{tr}(\rho(g))$ for all $g \in G$.

In this case $\dim V = \chi_{(V,\rho)}(1)$ and we sometimes call this number the *degree* of the character.

We say that $\chi_{(V,\rho)}$ is *irreducible* if (V, ρ) is an irreducible representation.

Let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G.

Remark 1.2. For any G-representations (V, ρ) and (V', ρ'), the following properties hold:

(1) If $(V, \rho) \cong (V', \rho')$ then $\chi_{(V,\rho)} = \chi_{(V',\rho')}$.

(2) The character $\chi = \chi_{(V,\rho)}$ is a *class function* on G, meaning that it is constant on conjugacy classes.

When $K[G]$ is semisimple, some additional properties hold:

(3) $\mathrm{Irr}(G)$ is a basis for the K-vector space of class functions $G \to K$.

(4) If $\mathrm{char}(K) = 0$, then $\chi_{(V,\rho)} = \chi_{(V',\rho')}$ if and only if $(V, \rho) \cong (V', \rho')$.

(5) $\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|$.

Example 1.3. If (V, ρ) is a G-representation with $\dim V = 1$, then $\chi_{(V,\rho)} = \rho$.

Example 1.4. Suppose $K = \mathbb{C}$ and G is a cyclic group of order $n \geq 1$ generated by an element x. Let χ_m be the map $\mathbb{C}[G] \to \mathbb{C}$ with $x^j \mapsto \zeta^{mj}$ where $\zeta = e^{2\pi\sqrt{-1}/n}$. Then $\mathrm{Irr}(G) = \{\chi_0, \chi_1, \chi_2, \dots, \chi_{n-1}\}$.

When a finite group is abelian (so that the group algebra is commutative), every irreducible representation is 1-dimensional. This is true of all commutative algebras.

Suppose V is a K-vector space. Write V^* for the vector space of linear maps $V \to K$.

If $f : V \to W$ is a linear map then let $f^* : W^* \to V^*$ be the linear map with $f^*(\lambda) = \lambda \circ f$.

If (V, ρ) is a G-representation, then its *dual representation* is (V^*, ρ^*) where $\rho^* : G \to \mathrm{GL}(V^*)$ is defined by the formula $\rho^*(g) = (\rho(g)^*)^{-1} = \rho(g^{-1})^*$ for $g \in G$.

Fact 1.5. When V is finite dimensional, we have $\chi_{V^*}(g) = \chi_V(g^{-1})$ for any $g \in G$. If we further assume that $K = \mathbb{C}$, then $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$ for all $g \in G$.

Suppose (V, ρ_V) and (W, ρ_W) are G-representations.

Their *tensor product* is the G-representation $(V \otimes W, \rho_{V\otimes W})$ where $\rho_{V\otimes W}(g)$ for $g \in G$ is the linear map $V \otimes W \to V \otimes W$ sending $v \otimes w \mapsto \rho_V(g)v \otimes \rho_W(g)w$ for all $v \in V$ and $w \in W$.

Fact 1.6. If $\dim V < \infty$ and $\dim W < \infty$ then $\chi_{(V\otimes W, \rho_{V\otimes W})} = \chi_{(V,\rho_V)}\chi_{(W,\rho_V)}$.

Remark 1.7. A G-representation is a left $K[G]$-module. The algebra $K[G]$ is often noncommutative.

Earlier, we emphasized that if A is a noncommutative algebra then the tensor product of two left A-modules is not a well-defined left A-module in general.

So how do we explain the existence of a tensor product for group representations?

Solution: the tensor product of two left A-modules does have the structure of a left $A \times A$-module. In particular, the tensor product of (V, ρ_V) and (W, ρ_W) is a representation of $K[G] \otimes K[G]$.

A special property of group algebras is that $K[G] \otimes K[G]$ has a subalgebra $K\text{-span}\{g \otimes g : g \in G\} \cong K[G]$.

By identifying $K[G]$ with this subalgebra, any $K[G] \otimes K[G]$-representation can be viewed as a $K[G]$-representation, and this is how we define the G-representation $(V, \rho_V) \otimes (W, \rho_W)$.

2 More special properties of characters

For the rest of today, we assume that G is a finite group.

Suppose V and W are G-representations. Let $\mathrm{Hom}_K(W, V)$ denote the set of linear maps $W \to V$.

The vector space $\mathrm{Hom}_K(W, V)$ is a left $K[G] \otimes K[G]$-module for the action

$$(g \otimes h) \cdot \varphi : w \mapsto g\varphi(h^{-1}w) \quad \text{for } g, h \in G.$$

Indeed, notice that if $\phi : W \to V$ is linear and $w \in W$ then

$$\big((g_1g_2 \otimes h_1h_2) \cdot \varphi\big)(w) = g_1g_2\varphi(h_2^{-1}h_1^{-1}w) = g_1\big(g_2 \otimes h_2 \cdot \varphi\big)(h_1^{-1}w) = \big((g_1 \otimes h_1)(g_2 \otimes h_2) \cdot \varphi\big)(w)$$

for any $g_1, g_2, h_1, h_2 \in G$. Now assume that V and W are finite-dimensional.

Proposition 2.1. It holds that $V \otimes W^* \cong \mathrm{Hom}_K(W, V)$ as $K[G] \otimes K[G]$-modules.

Proof. Let $F : V \otimes_K W^* \to \mathrm{Hom}_K(W, V)$ be the linear map sending

$$v \otimes \varphi \mapsto (w \mapsto \varphi(w)v) \quad \text{for } v \in V \text{ and } \varphi \in W^*.$$

Notice that if $\{v_i\}$ is a basis for V, $\{w_j\}$ is basis for W, and $\{\delta_j\}$ is the dual basis for W^*, then F sends $v_i \otimes \delta_j$ to the linear map $W \to V$ whose matrix in the chosen bases has a one in position (i, j) and zeros elsewhere. Any linear map $W \to V$ is a linear combination of such images $F(v_i \otimes \delta_j)$, so F is surjective.

Because $\dim(V \otimes_K W^*) = \dim V \dim W^* = \dim V \dim W = \dim(\mathrm{Hom}_K(W, V))$, as V and W are finite-dimensional, the map F is an isomorphism of K-vector spaces.

For any $g, h \in G$, $v \in V$, $w \in W$, and $\varphi \in W^*$, we have

$$\big((g \otimes h) \cdot F(v \otimes \varphi)\big)(w) = g\varphi(h^{-1}w)v$$

and

$$F\big((g \otimes h) \cdot (v \otimes \varphi)\big)(w) = F((gv) \otimes (\varphi \circ h^{-1}))(w) = \varphi(h^{-1}w)(gv) = g\varphi(h^{-1}w)v.$$

Hence, F is an isomorphism of $K[G] \otimes K[G]$-modules. □

By letting $g \in G$ act as $g \otimes g$, we can view $V \otimes_{K[G]} W^* \cong \mathrm{Hom}_K(W, V)$ as isomorphic G-modules.

Proposition 2.2. The set $(\mathrm{Hom}_K(W, V))^G$ of elements in $\mathrm{Hom}_K(W, V)$ fixed by all $g \in G$ is $\mathrm{Hom}_G(W, V)$.

Proof. Notice that if $\varphi \in \mathrm{Hom}_G(W, V)$, then for any $g \in G$, we have the following commutative diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"] & V \arrow[d, "g"] \\ V \arrow[r, "\varphi"] & V \end{tikzcd}.$$

Since the vertical map is invertible, we have $\varphi(w) = g(\varphi(g^{-1}w)) = (g \cdot \varphi)(w)$ for any $w \in W$.
Thus, $\mathrm{Hom}_G(W, V) \subseteq (\mathrm{Hom}_K(W, V))^G$.
Conversely, if $\varphi \in (\mathrm{Hom}_K(W, V))^G$, then for any $g \in G$ and $w \in W$, we have

$$\varphi(gw) = (g \cdot \varphi)(gw) = g\varphi(g^{-1}gw) = g\varphi(w).$$

Thus, $\varphi \in \mathrm{Hom}_G(W, V)$ and $(\mathrm{Hom}_K(W, V))^G \subseteq \mathrm{Hom}_G(W, V)$. $\square$

Combining the preceding results lets us deduce that:

Corollary 2.3. There is an isomorphism $(V \otimes_{K[G]} W^*)^G \cong \mathrm{Hom}_G(W, V)$ as G-modules.

From now on, we assume $K = \mathbb{C}$.
For any maps $f_1, f_2 : G \to \mathbb{C}$, we define a positive-definite Hermitian form

$$(f_1, f_2) := \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}.$$

Theorem 2.4. The set $\mathrm{Irr}(G)$ is an orthonormal basis for the class functions on G with respect to $(\cdot, \cdot)$.
In other words, we have $(\chi, \psi) = \delta_{\chi\psi}$ for any $\chi, \psi \in \mathrm{Irr}(G)$.

Proof. By Schur's Lemma, it suffices to prove that for any G-representations V and W, we have

$$(\chi_V, \chi_W) = \dim \mathrm{Hom}_G(W, V).$$

Let $\pi := \frac{1}{|G|} \sum_{g \in G} \in K[G]$. By Fact 1.5, we have

$$(\chi_V, \chi_W) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_W(g)} = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\chi_{W^*}(g) = \frac{1}{|G|} \sum_{g \in G} \chi_{V \otimes_{K[G]} W^*}(g) = \chi_{V \otimes_{K[G]} W^*}(\pi).$$

If X is any G-representation, then $X^G := \{x \in X : gx = x\}$ is a subrepresentation of G.
Notice that $g\pi = \frac{1}{|G|} \sum_{h \in G} gh = \frac{1}{|G|} \sum_{gh \in G} gh = \pi$ for any $g \in G$.
Therefore, we have $\pi x \in X^G$ for any $x \in X$ and $\pi : X \twoheadrightarrow X^G$ is a projection map.
Thus $\dim(X^G) = \chi_X(\pi)$. Restricting to the case when $X = V \otimes_{K[G]} W^*$, we get

$$\chi_{V \otimes_{K[G]} W^*}(\pi) = \dim(V \otimes_{K[G]} W^*)^G = \dim(\mathrm{Hom}_G(W, V))$$

by Corollary 2.3. $\square$

For $g \in G$, let $Z_g := \{h \in G : hgh^{-1} = g\}$ be the centralizer of g.
Also let $K_g := \{hgh^{-1} : h \in G\}$ be the conjugacy class of g.

Fact 2.5. By the *Orbit-Stabilizer Theorem* it holds that $|K_g| = \frac{|G|}{|Z_g|}$.

Theorem 2.6. Let $g, h \in G$. Then

$$\sum_{\psi \in \mathrm{Irr}(G)} \psi(g)\overline{\psi(h)} = \begin{cases} |Z_g| & K_g = K_h, \\ 0 & K_g \neq K_h. \end{cases}$$

Proof sketch. We want to describe this sum as the trace of a $\mathbb{C}$-endomorphism of $\mathbb{C}[G]$.

If we write V_ψ for an irreducible representation with character ψ, then we have

$$\begin{aligned} \sum_{\psi \in \mathrm{Irr}(G)} \psi(g)\overline{\psi(h)} &= \sum_{\psi \in \mathrm{Irr}(G)} \chi_{V_\psi}(g)\chi_{V_\psi^*}(h) \\ &= \sum_{\psi \in \mathrm{Irr}(G)} \chi_{V_\psi \otimes V_\psi^*}(g \otimes h) \\ &= \mathrm{tr}\big((\oplus_{\psi \in \mathrm{Irr}(G)} \rho_{V_\psi \otimes V_\psi^*})(g \otimes h)\big). \end{aligned}$$

We have an isomorphism $\bigoplus_{\psi \in \mathrm{Irr}(G)} V_\psi \otimes V_\psi^* \cong \oplus_{\psi \in \mathrm{Irr}(G)} \mathrm{End}(V_\psi) \cong \mathbb{C}[G]$ of $\mathbb{C}[G] \otimes \mathbb{C}[G]$ representations.

Under this isomorphism, $g \otimes h$ acts on $\mathbb{C}[G]$ as the linear map sending $x \in G$ to gxh^{-1}.

Thus $\sum_{\psi \in \mathrm{Irr}(G)} \psi(g)\overline{\psi(h)}$ is the trace of $x \mapsto gxh^{-1}$, which is

$$|\{x \in G : x = gxh^{-1}\}| = |\{x \in G : g = xhx^{-1}\}| = \begin{cases} |Z_g| & \text{if } K_g = K_h \\ 0 & \text{if } K_g \neq K_h. \end{cases}$$

□

3 Unitary representations

A finite-dimensional representation (V, ρ) of a group G (over $\mathbb{C}$) is *unitary* if there is a G-invariant positive definite Hermitian form $(\cdot, \cdot) : V \times V \to \mathbb{C}$ with

$$(\rho(g)v, \rho(g)w) = (v, w) \text{ for any } v, w \in V \text{ and } g \in G.$$

Proposition 3.1. If G is a finite group, then any finite dimensional G-representation is unitary.

Proof. Pick any basis $\{v_i\}$ for V. We define a positive-definite Hermitian form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ with

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Then the form $(v_i, v_j) := \sum_{g \in G} \langle gv_i, gv_j \rangle$ is positive-definite and Hermitian. □

Proposition 3.2. If (V, ρ) is a finite-dimensional unitary representation of a (not necessarily finite) group G , then (V, ρ) is semisimple.

Proof. Any irreducible representation is semisimple so assume V is reducible. Choose an irreducible subrepresentation of $U \subsetneq V$. Write $(\cdot, \cdot)$ for the form that makes V unitary.

Let $U^\perp = \{v \in V : (v, u) = 0 \text{ for all } u \in U\}$. Then $V = U \oplus U^\perp$ and $U^\perp$ is a subrepresentation since the relevant form is G-invariant, so the result follows by induction on dimension. □

4 Matrix elements

Continue to assume that G is a finite group and V is a finite dimensional irreducible $\mathbb{C}[G]$-module.

Choose a G-invariant positive definite Hermitian form $(\cdot,\cdot)$ on V and let $\{v_i\}$ be an orthonormal basis on V with respect to $(\cdot,\cdot)$. Define $t_{ij}^V(g) := (gv_i, v_j)$ for any $g \in G$.

For each pair (i,j) with $1 \leq i,j \leq \dim V$, the map $t_{ij}^V : G \to \mathbb{C}$ is a called a *matrix element*.

Proposition 4.1. The rescaled matrix elements $\frac{1}{\sqrt{\dim V}} t_{ij}^V : G \to \mathbb{C}$ (as V ranges over all isomorphism classes of finite dimensional irreducible G-representations and i,j range over the indices of an orthonormal basis of V) give an orthonormal basis of the space of *all* functions $G \to \mathbb{C}$.

We won't present the proof in class, but this can be found in the textbook.

Note that number of distinct matrix elements is $\sum_V (\dim V)^2 = |G|$.

4.1 Character tables

Suppose G is a finite group. Choose representatives $1 = g_1, g_2, \cdots, g_r$ for distinct conjugacy classes in G.

Suppose $\mathbf{1} = \chi_1, \chi_2, \cdots, \chi_r$ are the distinct elements in $\mathrm{Irr}(G)$.

Here $\mathbf{1}$ denotes the irreducible character $G \to \{1\}$.

Then everything you want to know about $\mathrm{Irr}(G)$ is encoded by the matrix

$\mathrm{Irr}(G)$	$1 = g_1$	g_2	$\cdots$	g_r
$\mathbf{1} = \chi_1$	1	1	$\cdots$	1
χ_2	$\chi_2(1)$	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
χ_r	$\chi_r(1)$	$\chi_r(g_2)$	$\cdots$	$\chi_(g_r)$

called the *character table* of G.

Example 4.2. If $G = S_3$, then the character table of G is

$\mathrm{Irr}(S_3)$	1	$(1,2)$	$(1,2,3)$
$1 = \chi_{(3)}$	1	1	1
$\chi_{(2,1)}$	2	0	-1
$\chi_{(1,1,1)}$	1	-1	1

Using the character table orthogonality relations from today, you can compute the sizes of all conjugacy classes in G. Then you can decompose arbitrary products of characters into irreducibles.